Attribution Linking: Proofed and Clarified

Over recent years, numerous methods have been developed to link single-period attribution results, with each creator presenting a rhetorical defense for their method. In a novel defense, this author uses mathematical proofs to illustrate how fundamental financial principles define linked attribution results.

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INTRODUCTION

The popular financial literature contains a great deal of debate (and some confusion) on the proper mathematics of performance and attribution. The debate has recently heated up in regards to linking single-period attribution results.[1] A recent article by Damien Laker (2002) describes a multi-period Brinson method for calculating cumulative attribution effects. Although this method does not produce results at the sector level,[2] the method produces portfolio level results.[3] Therefore, if the multi-period Brinson method calculates the "exact" portfolio level results, one could use this method to judge other more complete "approximate" methods. Nevertheless, before we use the multi-period Brinson method we need to challenge and/or verify the "exactness" of this methodology.

Although the analyst can often choose from a myriad of acceptable solutions to a particular challenge, at times it is inappropriate for the analyst to deviate from an accepted mathematical truth or axiom of finance. For example, while there are many methods for calculating single-period total returns,[4] the industry unanimously agrees on one universal method for compounding these single-period total returns. Similarly, while there are many methods for calculating single-period attribution effects,[5] there is only one mathematical truth for linking single-period attribution results (Frongello 2002). In the following paper I will:

1. provide sound mathematical proofs of cornerstone performance and attribution linking mathematics,
2. illustrate the importance of order dependence in cumulative performance and attribution results, and
3. analyze the multi-period Brinson linking methodology.

MATHEMATICAL PROCESS

One of the major impediments to resolution of any mathematical debate occurs when statements are made without adequate defense in the form of a mathematical proof.

For example, I could propose the statement, *If a right triangle has legs A and B and hypotenuse C*, then $\underline{A^2+B^2 = C^2}$. Now you may recognize this statement as the Pythagorean theorem, but, nevertheless, all I've given you is a statement containing a hypothesis (*italics)* and a conclusion (underlined), **but I have proved nothing**. A proper proof would consist of the steps leading from the hypothesis to the conclusion.[6] Then, the proof can only be challenged by challenging one of the steps. A formal proof is also very helpful in mapping the steps the mathematician took to solve the problem. For example, considering there are over 1000 valid proofs for the Pythagorean theorem, one would have trouble questioning the line of my logic without knowing the path I chose. Therefore, it would be helpful to provide mathematical proofs of the cornerstone performance and attribution linking mathematics. I will not reinvent any process in these proofs; each step in the logic of the following proofs will be supported by fundamental mathematical principles and truths. I will detail the mathematical proofs of compounded total return, compounded absolute attribution,[7] and compounded relative attribution.[8] Even if these forthcoming proofs do not give us any closure in the current debate, they will at least provide the steps for others to analyze and challenge.

COMPOUNDED RETURN

Statement

If the total returns of Period 1, 2, 3 .... n are represented by R_1, R_2, R_3 … R_n, then the cumulative return ($R_{1,n}$) over the Periods 1, 2, 3 … n equals:

$$R_{1,n} = [\prod_{i=1}^{n}(1+R_i)]-1 .$$

Proof

Assume you invest a certain beginning dollar base (B_1). The dollar return earned in Period 1 (D_1) equals the base of Period 1 (B_1) multiplied by the return of Period 1 (R_1).
Adding back the base, we get the total end amount of Period 1 (E_1), or, alternatively, the beginning amount of Period 2 (B_2).[9]

Where:

$D_1=B_1R_1$,
$E_1=B_1+B_1R_1$,
$E_1=B_1(1+R_1)$, and
$B_2=E_1=B_1(1+R_1)$.

In percentage terms, the return equals:

$E_1/B_1–1$,
$[B_1(1+R_1)]/B_1–1$,
$(1+R_1)/1–1$, and
R_1.

The return in Period 2 (R_2) is earned not only on the original base of Period 1 (B_1) but also on the dollars earned in Period 1 (D_1). In other words, the return of Period 2 (R_2) is earned on the base of Period 2 (B_2), which is equivalent to the end amount of Period 1 (E_1). Dollars earned in Period 2 (D_2) equals the base of Period 2 (B_2) multiplied by the return of Period 2 (R_2). Adding back the base of Period 2 (B_2), we arrive at the total end amount of Period 2 (E_2).

Where:

$D_2=B_2R_2$,
$E_2=B_2+B_2R_2$,
$E_2=B_2(1+R_2)$, and
$E_2=B_1(1+R_1)(1+R_2)$.

In dollar terms, the return earned over the two periods ($D_{1,2}$) equals the total dollars earned in Period 1 (D_1) plus the total dollars earned in Period 2 (D_2).

Where:

$D_{1,2}=D_1+D_2$,
$D_{1,2}=B_1R_1+B_2R_2$, and
$D_{1,2}=B_1R_1+B_1(1+R_1)R_2$.

In percentage terms, the cumulative return over Periods 1 and 2 ($R_{1,2}$) equals the end amount of Period 2 (E_2) divided by the base of Period 1 (B_1) minus 1.

$R_{1,2}=E_2/B_1–1$,
$R_{1,2}=[B_1(1+R_1)(1+R_2)]/B_1–1$, and
$R_{1,2}=[(1+R_1)(1+R_2)]–1$.

We can treat this result as a single-period return and repeat the process for additional periods. Substituting in $R_{1,2}$ for R_1 and R_3 for R_2 for Period 3, and so on and so forth for Periods R_4, R_5, … R_n, one is left with:

$$R_{1,n} = [\prod_{i=1}^{n}(1+R_i)]-1 ,$$

which completes the proof.

COMPOUNDED RETURN AND ORDER DEPENDENCE

Due to the commutative law of multiplication, which states that when multiplication is the only operation in an algebraic term, multiplication can be performed in any order without altering the result, ab = ba. In addition, the order of operations in algebraic expressions is as follows:

1. Parentheses/Brackets,

2. Multiplication/Division, and then

3. Addition/Subtraction.

This order must be followed, completing equivalent operations from left to right. Notice that the commutative law applies to our compound return formula. Because we are taking the product of the single-period returns within the parenthesis before we perform the subtraction of the one, the ordering of the periods will therefore have no bearing on the resulting product and cumulative total return. Alternatively, we can say that *compounded total returns are not order dependent*.

Consider an example. A portfolio manager earns 20% in Period 1 and 10% in Period 2 for a cumulative return of 32 percent. Now consider if the periods were reversed and the manager earned 10% in Period 1 and 20% in Period 2 for a cumulative return of 32 percent. Notice that the ordering of periods has no effect on the resulting cumulative total return.

CUMULATIVE ABSOLUTE ATTRIBUTION

Statement

If the total returns of Period 1, 2, 3 ... n are represented by R_1, R_2, R_3 ... R_n and the total return in any period is due to the sum of the absolute attributes A, B, C … N, then the cumulative return over Periods 1 to n due to a particular attribute Q ($Q_{1,n}$) is equal to:

$$Q_{1,n} = \sum_{i=1}^{n} [Q_i \prod_{j=1}^{i-1} (1+R_j)].$$

Proof

Assume you invest a certain beginning dollar base in Period 1 (B_1). The dollar return earned in Period 1 (D_1) equals the Period 1 base (B_1) multiplied by the Period 1 return (R_1).

Where:

$R_1=A_1+B_1+C_1+\ldots+N_1$,
$D_1=B_1R_1$, and
$D_1=B_1(A_1+B_1+C_1+\ldots N_1)$.

Adding back the base, we get the total end amount (E_1):

$E_1=B_1+B_1(A_1+B_1+C_1+\ldots N_1)$ and
$E_1=B_1(1+A_1+B_1+C_1+\ldots N_1)$.

In percentage terms, the return equals:

E_1/B_1-1,
$B_1(1+A_1+B_1+C_1+\ldots N_1)/B_1-1$,
$(1+A_1+B_1+C_1+\ldots N_1)/1-1$, and
$A_1+B_1+C_1+\ldots N_1$.

The return in Period 2 ($R_2=A_2+B_2+C_2+\ldots N_2$) is earned not only on the original base of Period 1(B_1) but also on the dollars earned in Period 1 (D_1). In other words, the return of Period 2 (R_2) is earned on the base of Period 2 (B_2) which is also equivalent to the end amount of Period 1 (E_1). Dollars earned in Period 2 (D_2) equals the base of Period 2 (B_2) multiplied by the return (R_2), or, alternatively, the attributes ($A_2+B_2+C_2+\ldots N_2$) of Period 2. Adding back the base of Period 2 (B_2) we arrive at the total end amount of Period 2 (E_2).

Where:

$D_2=B_2R_2$,
$D_2=B_2(A_2+B_2+C_2+\ldots N_2)$,
$E_2=B_2+B_2(A_2+B_2+C_2+\ldots N_2)$,
$E_2=B_2(1+A_2+B_2+C_2+\ldots N_2)$, and
$E_2=B_1(1+R_1)(1+A_2+B_2+C_2+\ldots N_2)$.

In dollar terms, the return earned over the two periods due to a particular attribute Q ($D_{Q1,2}$) equals the sum of the dollars earned in Period 1 due to attribute Q (D_{Q1}) and the dollars earned in Period 2 due to attribute Q (D_{Q2}).

Where:

$D_{Q1,2}=D_{Q1}+D_{Q2}$,
$D_{Q1,2}=B_1Q_1+B_2Q_2$, and
$D_{Q1,2}=B_1Q_1+B_1(1+R_1)Q_2$.

In percentage terms, the cumulative return over the two periods due to attribute Q ($Q_{1,2}$) equals the total dollars earned due to attribute Q ($D_{Q1,2}$) divided by the base of Period 1 (B_1).

Where:

$Q_{1,2}=D_{Q1,2}/B_1$,
$Q_{1,2}=(B_1Q_1+B_1(1+R_1)Q_2)/B_1$, and
$Q_{1,2}=Q_1+(1+R_1)Q_2$.

We can treat this result as a single-period result and repeat the process for additional periods. Substituting in $Q_{1,2}$ for Q_1, Q_3 for Q_2, and $R_{1,2}$ for R_1 for Period 3, and so on and so forth for R_4, R_5, … R_n, one is left with:

$$Q_{1,n}=\sum_{i=1}^{n}[Q_i\prod_{j=1}^{i-1}(1+R_j)],$$

which completes the proof.

CUMULATIVE ABSOLUTE ATTRIBUTION AND ORDER DEPENDENCE

From our preceding proof, we can offer the following intuitive interpretation. The portion of the total cumulative return due to a particular attribute is the sum of each period's attribute scaled by the cumulative return through the prior period. Notice that we complete the operations within the brackets before we sum these results over the periods. The contribution to cumulative total return of a particular single-period attribute is a function of two items:

1. the magnitude of the attribute and

2. the cumulative total return earned by the portfolio through the prior period.

Notice here that we have shown that any particular attribute is scaled by the cumulative total return and not necessarily by the cumulative total return associated with that particular attribute alone. Why? Because, as was shown earlier, an attribute is a return earned off a certain base. The base of the period in question relative to the base of Period 1 is a function of the cumulative total return through the prior period. Defining the growth of the base by one particular attribute alone can significantly misstate the base and result in an inaccurate contribution to cumulative attribution. Therefore, since any single-period attribute's contribution to cumulative return is partly dependent on the return through the prior period, or similarly the period in which the attribute occurs, it is clear that *cumulative absolute attribution is order dependent.*

Consider an example. A portfolio earns a 10% total return in Period 1 due entirely to attribute A. In Period 2, the portfolio earns a 10% total return due entirely to attribute B. Although the attribute B earned 10% in Period 2, the contribution to cumulative return is actually 11 percent. We arrive at the Period 2 contribution to cumulative return by scaling the attribute B by the Period 1 total return, regardless of whether the Period 1 return was earned by attribute A and/or B. In calculating the contribution to total cumulative return of attributes, we should be concerned with how much the original investment base has grown or, likewise, how much cumulative total return the portfolio has earned through the prior period. Attribution effects are earned off a base that may have grown due to any combination of attributes. Notice that we scaled the 10% Period 2 return due to attribute B, by the Period 1 total return of 10 percent. In scaling the Period 2 attribute we are only concerned with the prior cumulative total return and not where this total return came from. By scaling attributes in this manner, we relate the single-period results to their impact on cumulative return. We see from the example that attribute A contributed 10% and attribute B contributed 11% to the total return of 21 percent. If we reverse the periods, A contributes 11% and B contributes 10 percent. Thus, the ordering of periods changes the cumulative absolute attribution results, indicating the order dependence of cumulative absolute attribution.

CUMULATIVE RELATIVE ATTRIBUTION

Statement

Given that the difference between the portfolio return during a Period t, R_t, and the benchmark return during a Period t, $\overline{R}_t$ can be explained by the sum of a set of relative attributes a_t, b_t, c_t … n_t, where attribution effect G_{tb}

represents the relative outperformance due to attribute b during time t, then the cumulative relative outperformance due to attribute b between Period 1 and Period n ($b_{1,n}$) equals (Frongello, 2002):

$$b_{1,n} = \sum_{t=1}^{n} F_{tb},$$

where:

$$F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \overline{R}_t \sum_{j=1}^{t-1} F_{jb} .$$

Proof

Where:

$R_t - \overline{R}_t = a_t + b_t + c_t + \ldots n_t$ and
$R_t = \overline{R}_t + a_t + b_t + c_t + \ldots n_t$.

We saw in an earlier proof that the cumulative return for the portfolio over two periods ($R_{1,2}$) can be shown as:

$$
\begin{aligned}
R_{1,2} &= (1+R_1)(1+R_2)-1,\\
R_{1,2} &= (1+\overline{R}_1+a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2+a_2+b_2+c_2+\ldots n_2)-1,\\
R_{1,2} &= (1+\overline{R}_1+a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2)\\
&\quad +(1+\overline{R}_1+a_1+b_1+c_1+\ldots n_1)(a_2+b_2+c_2+\ldots n_2)-1,\\
R_{1,2} &= (1+\overline{R}_1)(1+\overline{R}_2)+(a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2)\\
&\quad +(1+\overline{R}_1+a_1+b_1+c_1+\ldots n_1)(a_2+b_2+c_2+\ldots n_2)-1,\\
R_{1,2} &= (1+\overline{R}_1)(1+\overline{R}_2)+(a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2)\\
&\quad +(1+R_1)(a_2+b_2+c_2+\ldots n_2)-1,\\
R_{1,2} &= (1+\overline{R}_1)(1+\overline{R}_2)-1+(a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2)\\
&\quad +(1+R_1)(a_2+b_2+c_2+\ldots n_2),\\
R_{1,2} &= \overline{R}_{1,2}+(a_1+b_1+c_1+\ldots n_1)(1+\overline{R}_2)+(1+R_1)(a_2+b_2+c_2+\ldots n_2),\\
R_{1,2} &= \overline{R}_{1,2}+(a_1+b_1+c_1+\ldots n_1)+\overline{R}_2(a_1+b_1+c_1+\ldots n_1)\\
&\quad +(1+R_1)(a_2+b_2+c_2+\ldots n_2), \text{ and}\\
R_{1,2}-\overline{R}_{1,2} &= (a_1+b_1+c_1+\ldots n_1)+(1+R_1)(a_2+b_2+c_2+\ldots n_2)\\
&\quad +\overline{R}_2(a_1+b_1+c_1+\ldots n_1).
\end{aligned}
$$

Since we know that the contribution to relative cumulative outperformance of Period 1 is equivalent to the sum of the attributes of Period 1 ($a_1+b_1+c_1+\ldots n_1$), then we can say the contribution of Period 2 to the total cumulative relative outperformance is equivalent to $(1+R_1)(a_2+b_2+c_2+\ldots n_2)+\overline{R}_2(a_1+b_1+c_1+\ldots n_1)$. Using the distributive property of multiplication, which states that if A(B+C+D) then AB+AC+AD, the cumulative effect of an attribute b over the two periods ($b_{1,2}$) can be shown as, $b_{1,2}=b_1+(1+R_1)b_2+\overline{R}_2 b_1$. We can treat this result as a single-period result and repeat the process for additional periods. Substituting in $b_{1,2}$ for b_1, $R_{1,2}$ for R_1, b_3 for b_2, and $\overline{R}_3$ for $\overline{R}_2$ in Period 3, and so on and so forth for Periods 4, 5, … n, we are left with the Frongello linking algorithm (2002).[10]

The Frongello Linking Algorithm:

$$b_{1,n} = \sum_{t=1}^{n} F_{tb},$$

where:

$$F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \overline{R}_t \sum_{j=1}^{t-1} F_{jb} ,$$

which completes the proof.

CUMULATIVE RELATIVE ATTRIBUTION AND ORDER DEPENDENCE

From the above proof, I can offer the following intuitive interpretation. The portion of the total cumulative relative return due to a particular attribute is the sum of a set of expressions. The expression in each period equals the period's attribute scaled by the cumulative return through the prior period plus the product of the cumulative attribution for that attribute through the prior period and the current period benchmark return. Again, notice that we complete the operations within the brackets before we sum these results for the periods. A particular period attribute's contribution to cumulative relative over/under performance is a function of three items:

1. the magnitude of the attribute,

2. the cumulative total performance earned by the portfolio prior to the period in which the particular attribute occurred, and

3. the rate of total return of the benchmark in the period in which the attribute occurred.

Therefore, since we have shown that the contribution to cumulative over/underperformance of a particular attribute is due not only to the magnitude of the particular attribute but also by the period in which it was earned, we can conclude that *cumulative relative attribution is order dependent.*

Consider another example. In Period 1, our portfolio returns 20% while the benchmark returns 10 percent. Our outperformance of 10% is due to 5% from allocation and 5% from selection. In Period 2, our portfolio returns 10% while the benchmark returns 5 percent. Our outperformance of 5% is due to 2% from allocation and 3% from selection. The cumulative performance of the portfolio is 32% versus the benchmark of 15.5% for an outperformance of 16.5 percent. We can explain this total from our attributes as follows. We add the Period 1 attributes to the scaled Period 2 attributes. Similar to absolute attribution, to scale relative attributes we take the product of the period attribute and the cumulative return through the prior period:

- Allocation =5%+2%(1+20%)=7.4% and
- Selection =5%+3%(1+20%)=8.6%.

At this point, we've applied the same scaling we used in the absolute compounding. However, our attributes only sum to 16% and we're trying to explain 16.5 percent. Where is this extra 0.5% coming from? While the absolute attribution we saw earlier explained all total portfolio performance by a set of attributes, relative attribution only explains the active return relative to a benchmark. We have to recognize in relative attribution that the total portfolio return is composed of the attributes and the benchmark return. So far, we've only addressed the scaling of the attributes. We have to recognize that in Period 2 part of our 16.5% cumulative outperformance occurs because the outperformance in Period 1 (our attributes in Period 1) is also earning the benchmark component of return. We simply recognize that a portion of our base (the portion that over/underperformed the benchmark in Period 1) is earning the benchmark component of return. To the scaled attributes, we add the product of the net prior effect due to each attribute and the current period benchmark return. Finally, we've explained 100% of the relative 16.5% performance.

- Allocation =7.4%+(5%x5%)=7.65% and
- Selection =8.6%+(5%x5%)=8.85%.

With periods reversed, the results become:

- Allocation =2%+5%(1+10%)+(2%x10%) =7.7% and
- Selection =3%+5%(1+10%)+(3%x10%) =8.8%.

Changing the ordering of periods produces very similar, but, nonetheless, different results. Similar to absolute attribution, properly compounded relative attribution is order dependent.

MATHEMATICAL CERTAINTY

The preceding proofs are not simply the opinion of an individual analyst which can be open to debate and interpretation. These proofs are the mathematical certainties of performance and attribution linking which cannot be ignored. The tools of my argument and defense are the unchallenged sound principles of mathematics. I'll embrace and appreciate anyone who can help me understand any potential flaw in my logic. My first comrade to question the logic presented in this paper is a gentleman by the name of Damien Laker, and the rest of my paper will address his comments.

THE MULTI-PERIOD BRINSON MODEL

The method described by Laker (2002) uses the portfolio quadrants (See Figure 1) described by Gary Brinson and Nimrod Fachler in 1985. The Brinson method is an industry accepted methodology for calculating single-period attribution results. As you can see in Figure 1, the single-period attributes are calculated simply by subtracting the appropriate quadrants. In the multi-period Brinson model described by Laker, the cumulative portfolio level attributes are computed similarly by compounding the single-period quadrant portfolios over multiple periods and performing the same arithmetic described in Figure 1. Laker makes the majority of the comments and criticisms in his paper based on the assumption that the multi-period Brinson model produces the correct cumulative portfolio level results. Therefore, the easiest method to address the majority of Laker's linking comments and criticisms is to focus on challenging the multi-period Brinson method.

THE MULTI-PERIOD BRINSON METHODOLOGY

Laker (2002) offers for defense of the multi-period Brinson model, the following statements:

1. the method's "single exact answer flows directly from the axioms of the Brinson model,"
2. "...the universally accepted principle that portfolio returns compound over time,"
3. "The multiple-period attributes calculated this way always sum exactly to the active return,"
4. "...the naïvely compounded attributes tend to consistently support the exact method," and
5. "I suggest that order dependence is actually a weakness ….rather than a strength."

Figure 1
Brinson Attribution.

	Single Period Brinson		Multi-Period Brinson	
	Portfolio Sector Returns	**Benchmark Sector Returns**	**Portfolio Sector Returns**	**Benchmark Sector Returns**
Portfolio Sector Weights	(IV) Portfolio $\sum_j w_j^p \cdot r_j^p$	(II) Active Asset Allocation Fund $\sum_j w_j^p \cdot r_j^b$	(IV) Portfolio $\prod_t (1+\sum_j w_j^p \cdot r_j^p) - 1$	(II) Active Asset Allocation Fund $\prod_t (1+\sum_j w_j^p \cdot r_j^b) - 1$
Benchmark Sector Weights	(III) Active Stock Selection Fund $\sum_j w_j^b \cdot r_j^p$	(I) Benchmark $\sum_j w_j^b \cdot r_j^b$	(III) Active Stock Selection Fund $\prod_t (1+\sum_j w_j^b \cdot r_j^p) - 1$	(I) Benchmark $\prod_t (1+\sum_j w_j^b \cdot r_j^b) - 1$

Asset Allocation = II-I
Selection = III-I
Interaction = IV-III-II+I
Total = IV-I

I will deal with each of these comments in turn.

The method's "*single exact answer flows directly from the axioms of the Brinson model.*" Other than this statement, Laker leaves no further defense as to why when the single-period notional portfolios are compounded, they reflect the "exact" portfolio level attribution results. "*When one subtracts the benchmark return from the Quadrant II return, one obtains a "pure" measure of the extent to which asset allocation decisions have added value relative to the benchmark – whether the calculation is over one period or multiple periods*." Again, Laker makes very bold statements with no reasoning as to why this will be accurate over multiple periods. Laker leaves no further defense or line of reasoning as to the compoundability of the Brinson notional portfolios over time. He simply assumes that since the single-period Brinson model notional portfolios are accurate at explaining single-period attribution and because total returns compound over time that it is appropriate to compound the single-period notional portfolios. This is a poor assumption, as we will see next.

I would argue that the Brinson single-period notional portfolios are accurate at calculating the attribution of outperformance over a certain base in the single-period. But what Laker overlooks is that the Brinson model decomposes the growth over the base of a performance period, and at the end of each period, the base for the next performance period is influenced by total return. As seen in our earlier proofs, the contribution to cumulative outperformance of a particular attribute is partly due to the total cumulative performance earned by the portfolio through the prior period, or alternatively, the magnitude of the total growth over the original Period 1 base. Selection and allocation effects in later periods are earned off a base that may have grown due to selection or allocation or probably both. To compound selection (allocation) effects only on the portion of the base that has grown due to selection (allocation) effects alone will leave cumulative selection (allocation) effects severely understated.[11] Note that by compounding the notional portfolios (which by design capture only the single-period relative return due to allocation or selection) Laker's multi-period Brinson method introduces this same understatement. Remember this important point, attributes are simply a component of total return. Total return (and its components) compound over total return, rather than the collective effects of allocation or selection alone.

Laker seems to feel that because of "...*the universally accepted principle that portfolio returns compound over time*" this warrants the compounding of the components (Brinson notional portfolios) of total portfolio returns. Although Laker leaves this careless assumption undefended, I'll offer two counter arguments. First, as we have seen in the earlier proofs, although total returns compound over time in the traditional sense of the term "compound," we have also seen that the linking of total return attributes is a slightly more complex process. If the simple compounding of attributes worked without leaving a substantially unexplained residual, you (I) would not have read (written) this paper. As a second counter argument, let me offer something from the pen of Laker himself, "*it is not mathematically justifiable to simply compound attributes over time,*" which is a method known as naïve compounding. Compounding the notional portfolios in the spirit the multi-period Brinson model suggests is not only unjustified, but, as mentioned earlier, this method will leave cumulative selection and allocation effects severely understated. An interesting question arises at this point if both allocation and selection are severely understated, doesn't some attribute have to be overstated? I address this concern next.

"*The multiple-period attributes calculated this way always sum exactly to the active return.*" Referring back to Figure 1, it seems intuitive that the compound of Quadrant IV portfolios will result in the cumulative return of the portfolio and that the compound of Quadrant I portfolios will result in the cumulative return of the benchmark. The difference in these will be the total value added, which we hope to describe as the sum of the portfolio level attributes. The asset allocation (security selection) in the single-period is defined as the portfolio in Quadrant II (III) minus the portfolio in Quadrant I. In the multi-period Brinson framework, the asset allocation (security selection) result is simply the compound return of all Quadrant II (III) portfolios minus the compound return of all Quadrant I portfolios. Again, this result will be severely understated because the compound of Quadrant II (III) portfolios will not capture the total cumulative effect of asset allocation (security selection). You'll notice that the final effect, interaction, is not calculated on its own, but rather it is the active return left over after allocation and selection are calculated. Therefore, regardless of how naïve the cumulative allocation and selection are calculated, the interaction serves as a fudge number that will always get you back to the total active return. As mentioned earlier, because selection and allocation will be severely understated, interaction will be overstated by an amount equal to the net understatement from allocation and selection. The degree of these misstatements will increase as the returns and periods under question increase.

Let's look at an example. Notice in Table 1, we have the single-period statistics from Frongello's 2002 Table 3. We have the single-period attribution calculated and the attribution also quoted in percent relative to the total outperformance.[12] Note that in the single-period, total allocation accounts for 37.5% of the relative outperformance, selection accounts for 62.5% of the relative outperformance, and interaction accounts for 0% of the relative outperformance at the portfolio level. Interestingly, when compounding identical periods of relative attribution, the proportional relationship of the attributes is maintained in the linked results. Why? Because with identical periods the single-period attribution results are identical, current attributes are scaled by the same amount (cumulative return through the prior period), and past attributes are carried into the future by the same amount (current period benchmark return), the proportionality holds. However, look at the Laker results after three periods. According to Laker, the percentage of relative outperformance due to allocation is 37.14%, selection is 62.14%, and interaction is 0.72 percent. At this point I must ask, how can three periods without any portfolio level interaction result in 0.72% of the cumulative outperformance due to interaction? As mentioned earlier, Laker is severely understating cumulative selection (allocation) results by compounding selection (allocation) results only off the portion of the base that has grown due to selection (allocation). This is wrong. As seen in our proof, any attribute earned in a period compounds over the cumulative total return through the prior period, regardless of how that cumulative total return was earned. As mentioned earlier, the misstatements become larger over more periods. For example, Table 2 shows the percentage of outperformance explained by each attribute when the Table 1 results are carried over an increasing number of periods. Notice how the Frongello, Cariño, and Menchero[13] methods maintain the proportionality of the results as the number of periods increases. However, when enough periods are linked

Table 1
A Re-Presentation of Frongello's Figure 3 (2002).

Single Period Statistics

	Portfolio		Benchmark	
	Weight	Return	Weight	Return
Stock	80.00%	6.00%	60.00%	5.00%
Bond	20.00%	3.00%	40.00%	2.00%
Total		5.40%		3.80%

Single Period Attribution

	Allocation	Selection	Interaction
Stock	0.24%	0.60%	0.20%
Bond	0.36%	0.40%	-0.20%
Total	0.60%	1.00%	0.00%
Difference	1.60%		

Single Period Attribution In Terms of Percent

	Allocation	Selection	Interaction
Stock	15.00%	37.50%	12.50%
Bond	22.50%	25.00%	-12.50%
Total	37.50%	62.50%	0.00%
Difference	100.00%		

Three Identical Periods Compounded

Portfolio	17.09%
Benchmark	11.84%
Difference	5.25%

Frongello/Cariño/Menchero

	Allocation	Selection	Interaction
Stock	0.79%	1.97%	0.66%
Bond	1.18%	1.31%	-0.66%
Total	1.97%	3.28%	0.00%
% of Attribution	37.50%	62.50%	0.00%

Multi-Period Brinson

	Allocation	Selection	Interaction
Stock	NA	NA	NA
Bond	NA	NA	NA
Total	1.95%	3.26%	0.04%
% of Attribution	37.14%	62.14%	0.72%

together with the multi-period Brinson method, the understatement for Selection and Allocation becomes so severe that ultimately 99.99% of the relative outperformance is said to come from interaction. I find this misleading, considering that each period has no interaction at all.

Let's consider another intuitive example that will bring to light the flaw in the multi-period Brinson method. Consider a small cap portfolio manager that decides the next month only to make selection bets in her portfolio. At the end of the month, she has earned a 40% return while the benchmark has returned only 10%, beating the index by 30% due entirely to selection. In the next month, she removes all selection bets and decides to make only allocation bets. At the end of the second month, she has once again beaten the index by 30%, achieving a total return of 40% while the index returns 10 percent. Over the two-month period, the manager has returned a total return of 96% while the index returns 21%, for an outperformance of 75 percent. The multi-period Brinson model would say that 33% comes from allocation, 33% comes from selection and the other 9% comes from interaction. How can 9% due to interaction make sense when neither period witnessed any interaction effect? It does not make sense, and I believe anyone who has read this far would agree. At this point, it is clear that allocation and selection are not being compounded properly in the multi-period Brinson model, and the model is using interaction to pick up the slack.

Table 2
Percent of Outperformance Explained By Attributes Over Time.

	Frongello/Cariño/Menchero			Multi-Period Brinson		
Periods	Allocation	Selection	Interaction	Allocation	Selection	Interaction
3	37.50%	62.50%	0.00%	37.14%	62.14%	0.72%
25	37.50%	62.50%	0.00%	33.27%	58.15%	8.58%
100	37.50%	62.50%	0.00%	21.56%	44.48%	33.97%
250	37.50%	62.50%	0.00%	7.20%	22.30%	70.50%
500	37.50%	62.50%	0.00%	0.80%	5.71%	93.48%
2000	37.50%	62.50%	0.00%	0.00%	0.00%	100.00%

"...The naïvely compounded attributes tend to consistently support the exact method." Laker noticed that the multi-period Brinson results in some situations seem to produce results quite similar to the naïve compounding of attributes. In all fairness, based on the naiveté of the multi Brinson Laker model…I wouldn't doubt this statement. You'll notice that the naïve method and Laker's multi-Brinson model make the same critical "naïve" mistake. Both methods compound attributes only on the portion of total return earned by that particular attribute alone. The only difference is that in the multi-period Brinson model the methodological misstatement that is usually left as an unexplained residual in the naïve method is conveniently buried in interaction. Due to the methods' similar critical flaw, it does not surprise me that the results are similar.

"*I suggest that order dependence is actually a weakness...rather than a strength. Compounding is a multiplicative process, and as everyone knows, ab = ba. This provides no reason whatsoever for thinking that it would be correct for a multi-period attribution method to produce different results if the order of periods was reversed.*" I agree with part of Laker's sentiment. Compounding is a multiplicative process, and I recognized the associative law of multiplication in my proof of cumulative total return when I showed that cumulative compound return is not order dependent. In my proofs, I showed that in mathematics it is necessary to respect not only the associative law of multiplication but also the order of operations governing mathematics. These sound principles of mathematics are neglected in the multi-period Brinson model. If multiplication were the only process involved in linking single-period attributes, then I would agree that the ordering of periods would not matter; however, attribution linking is a more dynamic mathematical process than the simple compounding of total returns. In properly linking attributes one has to recognize that single-period attributes are additive while the compounding is multiplicative, requiring that a specific sequence of multiplication and addition must be maintained in the proper linked results. As we have seen in earlier proofs, when computing compound total return, the results would not be affected by rearranging the periods. However, we defined both attribution-linking formulas as the sum of a set of expressions. These expressions must be completed before they are summed. Since the expressions scale the current attributes by the cumulative return earned through the prior period, the ordering of the periods will have an effect on the cumulative return through the prior period and thus, the scaling applied to the attributes. Contrary to Laker's comment, since we have shown that the ordering of the periods will have an impact on the scaling applied to the attributes, we can conclude that *properly linked attribution results are order dependent.*

CHALLENGE

At this point, I would like to address a challenge offered by Laker in regards to Table 3 of my latest paper. Laker asks the following question, "*If asset allocation subtracted value from the fund in the first two periods, and added 34% in Period 3, how can one possibly justify the conclusion that asset allocation*

Table 3
A Re-Presentation of Frongello's Figure 7 (2002).

Period 1	Portfolio Weight	Return	Benchmark Weight	Return		Allocation	Selection	Total
Stock	70.00%	45.00%	50.00%	-25.00%	Stock	-2.00%	49.00%	47.00%
Bond	30.00%	5.00%	50.00%	-5.00%	Bond	-2.00%	3.00%	1.00%
Total	100.00%	33.00%	100.00%	-15.00%	Total	-4.00%	52.00%	48.00%

Period 2	Portfolio Weight	Return	Benchmark Weight	Return		Allocation	Selection	Total
Stock	15.00%	40.00%	10.00%	10.00%	Stock	-0.45%	4.50%	4.05%
Bond	85.00%	40.00%	90.00%	20.00%	Bond	-0.05%	17.00%	16.95%
Total	100.00%	40.00%	100.00%	19.00%	Total	-0.50%	21.50%	21.00%

Period 3	Portfolio Weight	Return	Benchmark Weight	Return		Allocation	Selection	Total
Stock	95.00%	5.00%	10.00%	45.00%	Stock	30.60%	-38.00%	-7.40%
Bond	5.00%	45.00%	90.00%	5.00%	Bond	3.40%	2.00%	5.40%
Total	100.00%	7.00%	100.00%	9.00%	Total	34.00%	-36.00%	-2.00%

	Portfolio	Benchmark	Diff
Frongello	99.2340%	10.2535%	88.9805%

added substantially more than 34% over all three periods? I invite advocates of all the other methods to answer this specific question"

The correct answer for total allocation is 57.3948 percent. Here is how I arrived at this answer. First, keep in mind that we are trying to explain our overperformance of 88.9805% relative to the index due to the attributes allocation and selection. We can see plainly, that after Period 1, allocation was responsible for detracting –4% relative to the benchmark, while the portfolio overall returned 33% due to incredible stock selection. In Period 2, relative to the new investment base (the base in Period 2 is now 33% larger than the original Period 1 base), the portfolio earned –0.50% due to allocation. Still, relative to the original investment base, the –0.50% actually contributes –0.50%(1.33)= –0.665 percent. Also at this point, we have a slight challenge. Our goal is to describe the total outperformance in terms of the attributes. However, we need to recognize that in Period 2 the portfolio is earning more at the benchmark component of return. How? Unlike the first period, the larger second period base of the portfolio has a larger investment base in the benchmark. Alternatively, a portion of the base in Period 2, due to the attributes earned in Period 1, is earning the benchmark return. If our goal is to quote all outperformance in terms of attribution effects, then the component of outperformance due simply to having a higher base earning the benchmark return will not be picked up in the attributes without intervention. Recognize that the additional portion of the portfolio base that is earning the passive return in Period 2 is due to the attributes earned in Period 1. So in Period 2, we recognize that any attribute earned in Period 1, compounds into the future with the benchmark return. Thus, in Period 2, we also add the product of the return due to allocation in Period 1, –4%, and the benchmark return of Period 2, 19%, or –4%(19%) = –0.76 percent. This results in a total Period 2 contribution to relative return of –0.665% + –0.76% = –1.425 percent. Cumulative contribution to relative return due to allocation over Periods 1 and 2 then equals –4% + –1.425% = –5.425 percent. Period 3 is handled the same way, return due to allocation equals 34%, but keep in mind that this is the period return over a base that has grown (1 + 0.33)(1 + 0.40)–1=86.2 percent. So the Period 3 attribute of 34% equates to a cumulative contribution to relative return due to allocation of 34%(1.862) = 63.308 percent. Again the allocation results of the prior periods (–5.425%) are carried through the third period at the rate of return of the benchmark (9%), –5.425%*9% = –0.48825 percent. Therefore our total contribution to relative return in Period 3 is equal to –0.48825% + 63.308% = 62.819750 percent. The combined contribution of all three periods equals –4% + –1.425% + 62.819750% = 57.39475 percent. Even though selection was primarily responsible for increasing the investment base in Periods 1 and 2, *the quoted returns due to allocation in each period are earned off the total base of that period, regardless of how that larger base was achieved*. The total investment base grew by 86.2% right before the Period 3 allocation attribute of 34 percent. Similar to compounding, this nominal 34% relative single-period return equated to a contribution toward cumulative outperformance of 63% simply because the base grew by a large percent over Periods 1 and 2. We all need to recognize that single-period attribution results are relative to the base of the single-period and we need to relate these single-period results to the base of the entire cumulative period in order to understand attribution over the cumulative period.

CONCLUSION

The purpose of this paper was twofold,

1. to provide the mathematical proofs for cornerstone performance and attribution linking mathematics while illustrating the importance of order dependence, and

2. to critique the multi-period Brinson methodology.

While I recognize the various single-period return calculation and attribution schemes embraced as acceptable alternatives by the investment community, I maintain that the preceding proofs and formulae stand as the universal methods for compounding single-period returns and attribution statistics. The linking of performance and attribution results is not open to interpretation and/or debate. The methodologies provided in the preceding proofs should be all encompassing of any and every arithmetic single-period attribution scheme. These proofs not only outline the true mathematical relationship involved in linking these statistics but also bring to light the topic of order dependence. While compounded total returns are not order dependent, I showed that cumulative absolute and relative attribution results are order dependent.

The second goal of the paper served as a response and critique towards Laker's multi-period Brinson model. I have provided overwhelming evidence of the shortcomings of the multi-period Brinson model that I will recap briefly. First, before I even questioned the accuracy of the multi-period Brinson model's portfolio level results, I noted that a serious shortcoming of the multi-period Brinson method is that it is incapable of calculating sector level results. Despite this deficiency, Laker argued that the portfolio level results are "exact" and continued that the multi-period Brinson method should be used to judge or scale more complete "approximate" methods. I continued by challenging the accuracy of the "exact" portfolio level results of the multi-period Brinson methodology. I found that the multi-period Brinson method is critically flawed and offered the following observations as evidence. By compounding the active selection (allocation) single-period notional portfolios, the method only compounds the single-period selection (allocation) results earned over the portion of the portfolio that has grown due to selection (allocation) alone. This is a perilous flaw of the method. I've illustrated that because single-period attributes compound over the total cumulative return of the portfolio, the treatment described in the multi-period Brinson example will leave selection and allocation severely misstated with the interaction term serving as a fudge number. We've seen this with straightforward and intuitive examples. Furthermore, through the preceding examples and proofs I have established that although compound total returns are not order dependent, the accurate linking of single-period attribution results are order dependent. All in all, the multi-period Brinson method is a weak linking algorithm. It is not only inaccurate but it is also incomplete.

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ENDNOTES

[1] Singer (1998), Cariño (1999), Menchero (2000), Kirievsky and Kirievsky (2000), Davies and Laker (2001), Mirabelli (2000), Frongello (2002), Laker (2002).

[2] The multi-period Brinson method cannot calculate the cumulative selection effects within asset classes or the cumulative allocation effects within asset classes. For example, it cannot calculate the cumulative effect of stock selection, the cumulative effect of bond allocation, etc.

[3] The multi-period Brinson method produces a cumulative selection effect of all classes together and a cumulative allocation effect of all asset classes together.

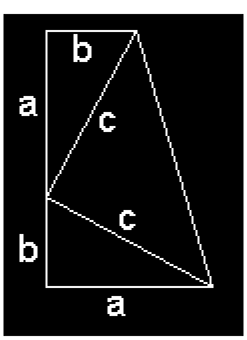
[4] Linked daily, linked between cash flow dates, Dietz, modified Dietz, internal rate of return, etc.

[5] Fama (1972), Brinson and Fachler (1985), Dietz, Fogler, Hardy (1980), Rudd and Clasing (1982), Fong, Pearson, and Vasicek (1983), Allen (1991), Ankrim (1992), Ankrim and Hensel (1992), Karnosky and Singer (1994), Burnie, Knowles and Teder (1998), Singer, Gonzalo, and Lederman (1998)

[6] An example of a simple proof of the Pythagorean theorem can be found at this website: http://asuwlink.uwyo.edu/~lane.

This proof was discovered by President James A. Garfield in 1876. The key of this proof is to use the formula for the area of a trapezoid : half sum of the bases times the altitude = (a + b)/2 * (a + b). We will put this formula for the area of the trapezoid on the left-hand side of the equation. We will then put the sum of the area of the three right triangles on the right-hand side of the equation. After simplification we will then get back to $a^2 + b^2 = c^2$.

$$\tfrac{1}{2}(a + b)(a + b) = \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}cc$$
$$\tfrac{1}{2}(a + b)^2 = ab + \tfrac{1}{2}c^2$$
$$(a + b)^2 = 2ab + c^2$$
$$a^2 + 2ab + b^2 = 2ab + c^2$$
$$a^2 + b^2 = c^2$$



[7] Absolute attribution is the decomposition of a portfolio total return among a set of attributes.

[8] Relative attribution is the decomposition of the over/underperformance of a portfolio relative to a benchmark.

[9] These proofs assume no cash flows. Because percentage returns are not affected by cash flows, the formulas hold. However, in actually calculating period returns amidst flows, one recommedation I would make is to value the securites on cash flow dates and compound sub periods between these dates to maintain accuracy. Note that in this situation the end dollar amount of the prior period will not agree to the beginning dollar amount of the current period however, percentage return will be left unaffected. I continue with this assumption throughout the paper.

[10] Damien Laker insists that the Frongello linking method (2002) was actually a globally accepted linking algorithm long before I wrote my paper in 2002. However, Damien Laker has been unable to produce any published sources outlining the Frongello method prior to my 2002 paper.

[11] The multi-period Brinson cumulative selection (allocation) will generally be underscaled, meaning less negative or less positive, when the selection (allocation) notional portfolio has a lower cumulative return than the cumulative return of the portfolio as a whole. For the rest of my paper, I will assume that the active manager is adding value in selection and allocation, but keep in mind that the misstatement can occur in the other direction depending on the relative performance of the compounded notional portfolios.

[12] I've chosen to illustrate the total attribution in percentage terms to help illustrate the distortion introduced by the multi-period Brinson method without requiring the reader to judge the appropriateness of nominal cumulative attributes relative to the nominal cumulative outperformance.

[13] Although the Frongello method is the only method for calculating true cumulative effects, I applaud Menchero and Cariño in developing accurate approximations. Only in the special case of identical duplicate periods do the Frongello, Cariño, and Menchero methods produce the same results. Frongello being the only order dependent method.