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Linking of Attributes Results

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INTRODUCTION

This chapter deals with the linking of single-period attribution results. Although the compounding of single-period total returns is rather straightforward, the linking of attribution effects introduces additional challenges. We will begin by looking at the two approaches for defining excess return, geometric and arithmetic. We will then proceed with a popular single-period attribution scheme called the Brinson–Fachler (BF, 1985) method. Next we will see how these single-period results are presented in the following geometric methods: the Burnie, Knowles and Teder (BKT, 1998), the BKT exponential, the pure geometric, the Cariño (1999) adjusted geometric and the Menchero (2005) adjusted geometric. We will see that the strength of geometric attribution lies in the ability to easily present multiple-period results. After the geometric presentations, we will compare and contrast the following arithmetic linking algorithms: Cariño, Menchero and Frongello.

RETURNS

Before we can address the challenge of linking attribution effects, it is necessary to define the excess return that the attributes will eventually explain. Intuitively, excess return is simply the performance of the portfolio relative to some benchmark performance. Money mangers aim for positive excess performance over the respective benchmark return. Mechanically, there are two approaches for calculating and presenting excess return. These include the arithmetic and geometric approaches.

Arithmetic excess return

The arithmetic excess return in a single period is the portfolio return minus the return of the benchmark

$$E^A = \text{arithmetic excess return}$$
$$R = \text{portfolio return}$$
$$\overline{R} = \text{benchmark return}$$
$$E^A = R - \overline{R}$$

Geometric excess return

The geometric excess return is the ratio of one plus the portfolio return divided by one plus the benchmark return minus one

$$E^G = \text{geometric excess return}$$
$$E^G = \frac{(1+R)}{\left(1+\overline{R}\right)} - 1$$

Single-period total return comparison

Due to the simple and immediate intuitive appeal of the arithmetic method, this is usually the preferred approach. The arithmetic approach answers the question: "What is the excess return relative to a base of zero?" On the other hand, the geometric approach answers the question: "What is the excess return relative to the level of the benchmark?". In other words, the arithmetic approach explains how much better you did than the benchmark as a percentage of the initial investment amount and the geometric approach explains how much better you did than the benchmark as a percentage of the final value of the initial amount invested in the benchmark. We will show the comparison of the presentation of the two methods with an example, see Table 1. Consider the arithmetic *versus* geometric example shown here. Which manager did better? It really depends on your definition of "better". Remember that the two approaches answer different questions; the benefit of the arithmetic presentation is that it is very intuitive to the audience. It is instinctive to simply subtract one return from the other to arrive at the relative performance. However, the geometric approach,

Table 1

	Manager A (%)	Manager B (%)
Portfolio	20.00	9.50
Benchmark	10.00	0.00
Arithmetic excess	10.00	9.50
Geometric excess	9.09	9.50

although foreign to some, offers some benefits. As outlined by Bacon,[1] these include the following.

Bacon's arguments for the geometric method:

1. *Proportional* – As explained earlier, the geometric approach measures excess return relative to the performance of the benchmark. It is much more impressive to have a positive excess return when the market performs poorly than during a time when the market performs well. In the example above, manager A outperformed B arithmetically. However, manager A was operating in a bullish investment space, while manager B outperformed by roughly the same arithmetic difference in a flat market. The geometric presentation takes this into account and portrays manager B as the outperforming manager.
2. *Convertible* – Let us say our managers are domestic managers in two countries A and B. Assume we measure manager performance in terms of currency B. What if market A was flat and all the benchmark performance is due to currency effects? After conversion to currency B, manager A's benchmark returned zero and the portfolio returned 9.09%. This is exactly the conclusion arrived at with the geometric approach. The relative ranking of the managers remains the same regardless of the currency quoted.
3. *Compoundable* – Geometric excess returns are easily compounded to arrive at multiple-period excess returns.[2] Arithmetic excess returns alone cannot. This leads to relatively easy geometric multiple-period analysis, more on this later.

Arithmetic versus *geometric attribution*

Attribution effects explain the total excess return relative to a benchmark. In arithmetic attribution, these effects add to the total

Table 2

	Portfolio (%)	Benchmark (%)	Arith. diff. (%)	Geo. diff. (%)
Period 1	3.93	1.88	2.05	2.01

excess return. In geometric attribution, these effects compound to the total excess return.

Single-period returns

The single-period returns given in Table 2 will be used in the examples to follow.

BF attribution

The arithmetic excess return is commonly decomposed by way of the BF (1985) method. To begin we need to introduce the following variables and formulas of the additive BF scheme[3,4]

$$
\begin{aligned}
w_{it} &= \text{portfolio weight in sector } i \text{ in period } t \\
\overline{w}_{it} &= \text{benchmark weight in sector } i \text{ in period } t \\
r_{it} &= \text{portfolio return in sector } i \text{ in period } t \\
\overline{r}_{it} &= \text{benchmark return in sector } i \text{ in period } t \\
R_t &= \text{portfolio return in period } t \\
\overline{R}_t &= \text{benchmark return in period } t \\
A_{it}^{BF} &= \text{allocation effect in sector } i \text{ in period } t \\
A_t^{BF} &= \text{total allocation effect in period } t \\
S_{it}^{BF} &= \text{selection effect in sector } i \text{ in period } t \\
S_t^{BF} &= \text{total selection effect in period } t
\end{aligned}
$$

$$
\begin{aligned}
R_t &= \sum_i w_{it} r_{it} \\
\overline{R}_t &= \sum_i \overline{w}_{it} \overline{r}_{it} \\
A_{it}^{BF} &= \left(w_{it} - \overline{w}_{it}\right)\left(\overline{r}_{it} - \overline{R}_t\right) \\
A_t^{BF} &= \sum_i \left(w_{it} - \overline{w}_{it}\right)\left(\overline{r}_{it} - \overline{R}_t\right)
\end{aligned}
$$

Table 3

Period 1	**Portfolio**		**Benchmark**		**Attribution**	
	Weight (%)	**Return (%)**	**Weight (%)**	**Return (%)**	**Allocation (%)**	**Selection (%)**
Equity	15.00	12.00	25.00	9.00	−0.71	0.45
Bond	85.00	2.50	75.00	−0.50	−0.24	2.55
Total	100.00	3.93	100.00	1.88	−0.95	3.00

$$S_{it}^{BF} = (w_{it})(r_{it} - \overline{r}_{it})$$
$$S_{t}^{BF} = \sum_i (w_{it})(r_{it} - \overline{r}_{it})$$
$$E_t^A = A_t^{BF} + S_t^{BF}$$

The BF (1985) example is shown in Table 3. Here we see possible portfolio and benchmark compositions that agree with our total returns in Table 2. Note that the total allocation and selection effects sum to the total arithmetic excess return. Work it out! We will stay with these same figures throughout the examples to follow. In the following sections we will take these additive effects and fit them into a geometric context. If successful, these geometric equivalents will compound to the geometric excess return.

BKT geometric

Interestingly, the geometric methods also rely heavily on the arithmetic BF (1985) principles in one way or another. Below we illustrate the single-period geometric presentation attributed to BKT (1998)

$$A_{it}^{BKT} = \frac{A_{it}^{BF}}{1 + \sum_i w_{it}\overline{r}_{it}}$$
$$A_t^{BKT} = \sum_i A_{it}^{BKT}$$
$$S_{it}^{BKT} = \frac{S_{it}^{BF}}{1 + \sum_i w_{it}\overline{r}_{it}}$$
$$S_t^{BKT} = \sum_i S_{it}^{BKT}$$
$$E_t^G = \left(1 + A_t^{BKT}\right)\left(1 + S_t^{BKT}\right) - 1$$

Table 4 ■■■

Period 1	Allocation (%)	Selection (%)
Equity	−0.6994	0.4459
Bond	−0.2331	2.5266
Total	−0.9325	2.9725

Table 5 ■■■

Period 1	Allocation (%)	Selection (%)
Equity	−0.6901	0.4383
Bond	−0.2305	2.5092
Total	−0.9190	2.9585

The BKT (1998) geometric example is shown in Table 4. In the BKT geometric model, the additive BF (1985) attributes are divided by one plus the return of the notional portfolio. Here, the notional portfolio is defined as a hypothetical portfolio composed of portfolio weights and benchmark returns. After the transformation, the total allocation and selection effects are found by summing the respective sector level attributes. These totals then compound to the total geometric excess return for the period.

BKT exponential

In the original BKT approach, we note that the effects across sectors *sum* to their respective attribute totals. For those that prefer a multiplicative approach across all levels, Cariño (1999) proposed the following exponential adjustment to the BF (1985) arithmetic attributes

$$A_{it}^{EXP} = \exp\left[\left(\frac{\ln(1+R_t) - \ln(1+\bar{R}_t)}{R_t - \bar{R}_t}\right) A_{it}^{BF}\right] - 1$$

$$A_t^{EXP} = \left[\prod_i \left(1 + A_{it}^{EXP}\right)\right] - 1$$

$$S_{it}^{EXP} = \exp\left[\left(\frac{\ln(1+R_t) - \ln(1+\bar{R}_t)}{R_t - \bar{R}_t}\right) S_{it}^{BF}\right] - 1$$

$$S_t^{EXP} = \left[\prod_i \left(1 + S_{it}^{EXP}\right)\right] - 1$$

$$E_t^G = \left(1 + A_t^{EXP}\right)\left(1 + S_t^{EXP}\right) - 1$$

The BKT exponential example is given in Table 5. This approach is very similar to the original BKT model. However, in the exponential model, the single-period sector level allocation and selection effects compound to their attribute totals, instead of summing to the attribute totals as in the BKT model. Similar to the BKT model, these attribute totals compound to the total geometric excess return.

Pure geometric method

The previous two methods take the original BF attributes[5] and adjust them so they aggregate multiplicatively. In the pure geometric method, we attempt to find the geometric equivalent to the original arithmetic BF attributes. Instead of adjusting the arithmetically calculated attributes so that they behave geometrically, we want to calculate our attribution effects geometrically to begin with. They are found in the following manner

$$A_{it}^{PUR} = \frac{1+(w_{it}-\bar{w}_{it})\bar{r}_{it}}{1+(w_{it}-\bar{w}_{it})\bar{R}_t} - 1$$

$$A_t^{PUR} \approx \left[\prod_i \left(1 + A_{it}^{PUR}\right)\right] - 1$$

$$S_{it}^{PUR} = \frac{1+w_{it}r_{it}}{1+w_{it}\bar{r}_{it}} - 1$$

$$S_t^{PUR} \approx \left[\prod_i \left(1 + S_{it}^{PUR}\right)\right] - 1$$

$$E_t^G \approx \left(1 + A_t^{PUR}\right)\left(1 + S_t^{PUR}\right) - 1$$

The pure geometric example is given in Table 6. Note the approximation symbols in the formulas. The attribute totals are only approximates and the total geometric excess return is not exactly explained by compounding the attribute totals. Although the pure geometric approach provides close approximates, this approach will never provide a complete explanation of total excess return without some distortion. This is an unavoidable consequence of the pure geometric method. However, as we will see, adjustments have been proposed to solve this shortcoming.

Table 6

Period 1	Allocation (%)	Selection (%)
Equity	−0.7138	0.4440
Bond	−0.2371	2.5609
Total	−0.9492	3.0163

Table 7

Period 1	Allocation (%)	Selection (%)
Equity	−0.7048	0.4384
Bond	−0.2340	2.5279
Total	−0.9372	2.9774

Cariño adjusted pure geometric

Cariño (1999) engineered the following adjustment to resolve the pure geometric compounding problem

$$A_{it}^{CAR} = \left(1 + A_{it}^{PUR}\right) \frac{\ln\left[\left(1+R_t\right) / \left(1+\bar{R}_t\right)\right]}{\ln\left[\prod_i \left(1+S_{it}^{PUR}\right)\left(1+A_{it}^{PUR}\right)\right]} - 1$$

$$A_t^{CAR} = \left[\prod_i \left(1 + A_{it}^{CAR}\right)\right] - 1$$

$$S_{it}^{CAR} = \left(1 + S_{it}^{PUR}\right) \frac{\ln\left[\left(1+R_t\right) / \left(1+\bar{R}_t\right)\right]}{\ln\left[\prod_i \left(1+S_{it}^{PUR}\right)\left(1+A_{it}^{PUR}\right)\right]} - 1$$

$$S_t^{CAR} = \left[\prod_i \left(1 + S_{it}^{CAR}\right)\right] - 1$$

$$E_t^{G} = \left(1 + A_t^{CAR}\right)\left(1 + S_t^{CAR}\right) - 1$$

The Cariño (1999) adjusted pure geometric example is given in Table 7. With the Cariño adjustment, all single-period results are multiplicative. Sector level attributes compound to the attribute totals and the attribute totals compound to the total geometric excess return.

Menchero adjusted pure geometric

Menchero (2005) argues that the Cariño pure geometric adjustments result in unjustifiably large deviations from the pure geometric results. Menchero proposes the following adjustments that minimise the corrections made

Table 8 ■■■

Period 1	Allocation (%)	Selection (%)
Equity	−0.7157	0.4433
Bond	−0.2373	2.5374
Total	−0.9512	2.9920

$$A_{it}^{MEN} = (1 + A_{it}^{PUR}) \times \exp\left[\left(\frac{\ln(1+R_t) - \ln(1+\bar{R}_t) - \sum_i \ln\left[(1+A_{it}^{PUR})(1+S_{it}^{PUR})\right]}{\sum_i \ln^2(1+A_{it}^{PUR}) + \sum_i \ln^2(1+S_{it}^{PUR})}\right)\ln^2(1+A_{it}^{PUR})\right] - 1$$

$$A_t^{MEN} = \left[\prod_i (1+A_{it}^{MEN})\right] - 1$$

$$S_{it}^{MEN} = (1 + S_{it}^{PUR}) \times \exp\left[\left(\frac{\ln(1+R_t) - \ln(1+\bar{R}_t) - \sum_i \ln\left[(1+A_{it}^{PUR})(1+S_{it}^{PUR})\right]}{\sum_i \ln^2(1+A_{it}^{PUR}) + \sum_i \ln^2(1+S_{it}^{PUR})}\right)\ln^2(1+S_{it}^{PUR})\right] - 1$$

$$S_t^{MEN} = \left[\prod_i (1+S_{it}^{MEN})\right] - 1$$

$$E_t^G = (1 + A_t^{MEN})(1 + S_t^{MEN}) - 1$$

The Menchero adjusted pure geometric example is given in Table 8. This method has the same benefits as the Cariño method in that the results are fully compoundable to arrive at the geometric excess return. Compared to the pure geometric attributes, you may notice that the Menchero adjusted numbers have smaller corrections than those produced by the Cariño adjustment.

Multiple-period geometric attribution

The strength of geometric attribution is the ease with which it handles multiple-period attribution. But as we will see, some geometric methods do not actually compound properly. To demonstrate this we will add two additional periods of analysis to the data we have been working with, see Table 9.

These total returns are broken down into the BF additive attributes, see Table 10. First, we will look at the BKT and the BKT exponential single- and multiple-period results, shown in Table 11.

Table 9

	Portfolio (%)	Benchmark (%)	Arith. diff. (%)	Geo. diff. (%)
Period 1	3.93	1.88	2.05	2.01
Period 2	2.40	4.83	−2.43	−2.31
Period 3	0.45	−3.70	4.15	4.31
Total	6.90	2.84	4.06	3.95

Table 10

	Portfolio		Benchmark		Attribution	
	Weight (%)	Return (%)	Weight (%)	Return (%)	Allocation (%)	Selection (%)
Period 1						
Equity	15.00	12.00	25.00	9.00	−0.71	0.45
Bond	85.00	2.50	75.00	−0.50	−0.24	2.55
Total	100.00	3.93	100.00	1.88	−0.95	3.00
Period 2						
Equity	80.00	4.50	65.00	8.50	0.55	−3.20
Bond	20.00	−6.00	35.00	−2.00	1.02	−0.80
Total	100.00	2.40	100.00	4.83	1.58	−4.00
Period 3						
Equity	30.00	1.50	20.00	−2.50	0.12	1.20
Bond	70.00	0.00	80.00	−4.00	0.03	2.80
Total	100.00	0.45	100.00	−3.70	0.15	4.00

In the BKT method, only the single-period attribute totals compound to multiple-period totals. These multiple-period attribute totals explain all of the excess return. Unfortunately, in the multiple-period context, the single-period sector level attributes do not compound to produce multiple-period sector totals. The multiple-period compounded sector level results do not agree exactly with the attribute totals and result in an unavoidable unexplained residual.

Cariño's exponential adjustment solves this shortfall. In the BKT exponential results, every single-period sector level attribute compounds to support the exact amount of the single-period excess return and ultimately the multiple-period totals as well.

Table 11

	BKT		BKT exponential	
	Allocation (%)	**Selection (%)**	**Allocation (%)**	**Selection (%)**
Period 1				
Equity	−0.6994	0.4459	−0.6901	0.4383
Bond	−0.2331	2.5266	−0.2305	2.5092
Total	−0.9325	2.9725	−0.9190	2.9585
Period 2				
Equity	0.5259	−3.0075	0.5335	−3.0414
Bond	0.9766	−0.7519	0.9930	−0.7692
Total	1.5025	−3.7594	1.5318	−3.7871
Period 3				
Equity	0.1246	1.2442	0.1221	1.2275
Bond	0.0312	2.9031	0.0305	2.8876
Total	0.1558	4.1472	0.1526	4.1505
Period 1–3				
Equity	−0.0528	−1.3629	−0.0384	−1.4210
Bond	0.7726	4.7098	0.7909	4.6580
Total	0.7126	3.2113	0.7522	3.1707
Excess	3.9468		3.9468	
Residual*	0.0000		0.0000	

*No residual at total level only, sector attributes do not compound.

One debatable criticism[6] of the BKT method is that the transformation of the additive BF attributes is biased by the return of the notional portfolio. As can be seen in the formula, a larger denominator in a period of larger returns will result in a lower adjustment and *vice versa*. This bias carries through into the BKT exponential adjustment. The BKT exponential adjustment contains an additional similar bias. Note that the term in parenthesis for the sector level effects is small when the portfolio and benchmark returns are large, and *vice versa*. As a consequence, attributes of the same magnitude in different periods will be scaled differently when return levels are different in those periods.

Next, we look at the pure geometric methods in Table 12. We could have anticipated that the pure geometric method would not produce residual free multiple-period attribution, because the pure geometric method did not explain the excess return in a single period. Compounding periods only exaggerates the unexplained

Table 12 ■■■

	Pure geometric		Adj. pure − Cariño		Adj. pure − Menchero	
	Allocation (%)	Selection (%)	Allocation (%)	Selection (%)	Allocation (%)	Selection (%)
Period 1						
Equity	−0.7138	0.4440	−0.7048	0.4384	−0.7157	0.4433
Bond	−0.2371	2.5609	−0.2340	2.5279	−0.2373	2.5374
Total	−0.9492	3.0163	−0.9372	2.9774	−0.9512	2.9920
Period 2						
Equity	0.5473	−2.9963	0.5627	−3.0789	0.5456	−3.0471
Bond	1.0312	−0.8032	1.0602	−0.8256	1.0252	−0.8069
Total	1.5841	−3.7754	1.6289	−3.8791	1.5764	−3.8294
Period 3						
Equity	0.1204	1.2091	0.1212	1.2169	0.1205	1.2132
Bond	0.0299	2.8807	0.0301	2.8996	0.0299	2.9043
Total	0.1504	4.1246	0.1513	4.1518	0.1504	4.1528
Period 1–3						
Equity	−0.0502	−1.3875	−0.0251	−1.4694	−0.0537	−1.4358
Bond	0.8218	4.6678	0.8541	4.6298	0.8156	4.6640
Total	0.7712	3.2155	0.8288	3.0924	0.7615	3.1612
Excess	4.0115		3.9468		3.9468	
Residual	0.0647		0.0000		0.0000	

distortion. Nevertheless, many feel that the strength of the pure geometric approach lies in the fact that the attributes are calculated geometrically and thus provide a better basis for geometric computations. Furthermore, because the residual is relatively small, efforts have been made to eliminate it.

The Cariño adjusted pure geometric method solves this problem. As we saw earlier, the Cariño adjustment allows for the compoundability of the attributes to the total excess return in the single period. Interestingly, the same compoundability holds at the multiple-period level when attributes are aggregated over multiple periods.

The Menchero adjusted pure geometric method contains the same benefits as the Cariño adjusted pure geometric method with one small difference. Menchero claims that his proposed adjustments provide smaller deviations from the pure geometric method than the deviations encountered with the Cariño method. From our example it appears that this is the case in some, but not all attributes.

Challenges of geometric attribution

After single-period geometric attributes are computed, they can often be compounded into the future with ease. However, it is not always easy getting to the adjusted single-period attributes. Nevertheless, the strength of geometric attribution is a strong selling point. We mentioned earlier that geometric attribution also has the benefit of being convertible and proportionate. Despite these benefits, the geometric method is not the popular method of presentation. To review, geometric attribution explains the difference in excess performance (expressed as a ratio) through a set of geometric attributes that compound to the total. Despite some of the benefits of looking at attribution in this way, many are simply not comfortable adapting to this mindset. Furthermore, the geometric methods we have seen are very tightly committed to the BF equity style approach and it is quite uncertain how one would fit elements such as currency and fixed income effects into the analysis if these sorts of bets are taken in the portfolio. Fortunately, these questions are easily addressed in the next section where we will look at the linking of arithmetic attributes.

SINGLE-PERIOD ARITHMETIC ATTRIBUTION

In the United States it is generally considered more intuitive, more natural and more traditional to think of excess return as a *difference*.[7] They are also most comfortable explaining that difference with a set of attributes that *add* to the difference. We have already seen an example of single-period arithmetic attribution. The BF method takes an additive difference in performance and explains that difference with a set of attributes that sum to that difference. We spent a lot of effort in the last section adjusting the BF effects that would link properly. Even after all that work the question was left unanswered: "What about fixed income or currency effects?" After all, there are many available methods of calculating these effects in the single period, but how would we force them into a geometric framework? If the single-period attribution includes effects in addition to those described earlier, then the analyst would be required to invent additional geometric formulas to accommodate these effects.

On the contrary, the beauty of the arithmetic linking algorithms presented in this section is that the single-period scheme is irrelevant as long as the single-period attributes add to the single-period excess return. Unlike the geometric methods, there is no added complexity to the analysis of fixed income and/or currency exposed portfolios.[8] As long as the attributes sum to the difference in excess return it really does not matter what the effects are or how they are calculated.

The linking problem

In an earlier section we saw the computation of BF attributes. We will use those numbers, but bear in mind that we could be using any number of attributes, regardless of how they are calculated, in our analysis. Also, we will keep the illustration simple for now and only focus on attribute totals; however, the same analysis can be performed on sector level attributes as well.

NAÏVE METHODS

On the surface, it may appear reasonable to compound or sum the attributes over the periods. This leads to a small residual as seen in the example given in Table 13.

Table 13

	Portfolio (%)	Benchmark (%)	Diff. (%)	Allocation (%)	Selection (%)	
Period 1	3.93	1.88	2.05	−0.95	3.00	
Period 2	2.40	4.83	−2.43	1.58	−4.00	
Period 3	0.45	−3.70	4.15	0.15	4.00	
Total	6.90	2.84	**4.06**			
						Total
			Sum	0.78	3.00	**3.78**
			Product	0.76	2.84	**3.62**

Desirable linking algorithm characteristics

Numerous methods have been proposed over the years on how to deal with this challenge. It is helpful to present a set of desirable characteristics to help separate the strong algorithms from the weaker ones. The best contribution to this set of criteria was first offered by Cariño (1999). We will present them next.

Generality

The methodology should support any additive single-period scheme.

You should not be bound by your linking algorithm when calculating single-period arithmetic attribution. The linking algorithm should work with any set of attributes as long as they sum to the total excess return. The analyst should not have to redefine the single-period formulas simply for the purpose of linking. For example, aside from any other limitations of the linking algorithm presented by Mirabelli (2000–2001), this method only links single-period equity based BF style attribution. Due to this dependency, the method could not be used to link any other single-period decomposition.

Familiarity

The interpretation of the multiple-period results should be the same as the interpretation of the single-period results.

In other words, the layout and presentation of the multiple-period results should mirror those of the single-period report. After linking, the analyst should not have to adapt to a different format of presentation. For example, the linking algorithm suggested by Laker (2002) would only report portfolio level allocation and selection results. Even if single-period sector level attribution effects were available, they were sacrificed in the linking process.

No residuals/distortion
The methodology should explain the excess return exactly without introducing unnecessary distortion.

If all single-period results do not have any residual, then all of the performance in all single periods is accounted for. It is a logical conclusion from here that there should not be any unaccounted performance in the multiple-period analysis. Other distortions are equally undesirable. A good example of an unwanted distortion occurs in the Campisi method. Here there are occasions when the algorithm can erroneously cause the sign of the attributes to switch. Negative attributes become positive and positive attributes become negative. Any distortion that causes deviation from reality should be avoided.

The algorithms that satisfy these basic criteria are the coefficient methods of Cariño and Menchero and the recursive Frongello (2002) method.

Coefficient methods
The idea behind a coefficient method is quite simple. Arithmetic attributes add to the excess return of the single period, however they cannot be summed or compounded to explain the total excess return over multiple periods

G_{tb} = original arithmetic attribute b in time t

R = cumulative portfolio return

$\bar{R}$ = cumulative benchmark return

$$\sum_b G_{tb} = R_t - \bar{R}_t$$

$$\sum_t \sum_b G_{tb} \neq R - \bar{R}$$

$$\left[\prod_t \prod_b (1 + G_{tb})\right] - 1 \neq R - \bar{R}$$

To solve this problem the original arithmetic attribute is multiplied by a scaling coefficient for that period. After all single-period original attributes have been transformed, the adjusted attributes sum to the total excess return over the periods

$$F_{tb} = \text{adjusted arithmetic attribute } b \text{ in time } t$$
$$C_t = \text{scaling factor in period } t$$
$$F_{tb} = G_{tb}(C_t)$$
$$\sum_t \sum_b F_{tb} = R - \bar{R}$$

The only challenge left is to calculate the single-period scaling coefficients.

The Cariño scaling coefficient

$$C_t^{CAR} = \frac{[\ln(1+R_t) - \ln(1+\bar{R}_t)]/(R_t - \bar{R}_t)}{[\ln(1+R) - \ln(1+\bar{R})]/(R - \bar{R})}$$

The Cariño example is shown in Tables 14 and 15.

The Menchero scaling coefficient

$T =$ total number of periods

$$C_t^{MEN} = (1/T)[(R-\bar{R})/((1+R)^{1/T} - (1+\bar{R})^{1/T})]$$
$$+ \frac{\left(R - \bar{R} - (1/T)[(R-\bar{R})/((1+R)^{1/T} - (1+\bar{R})^{1/T})]\sum_t (R_t - \bar{R}_t)\right)(R_t - \bar{R}_t)}{\sum_t (R_t - \bar{R}_t)^2}$$

The Menchero example is given in Tables 16 and 17.

The Frongello linking algorithm

$$F_{tb} = G_{tb} \cdot 0.5\left[\prod_{j=1}^{t-1}(1+R_j) + \prod_{j=1}^{t-1}(1+\bar{R}_j)\right] + 0.5(R_t + \bar{R}_t)\sum_{j=1}^{t-1} F_{jb}$$

This formula (modified Frongello) can be simplified if we allow the assumption that the portfolio return will roughly track the

Table 14

	Portfolio (%)	Benchmark (%)	Arith. diff. (%)	Cariño coef. (%)
Period 1	3.93	1.88	2.05	1.0190
Period 2	2.40	4.83	−2.43	1.0120
Period 3	0.45	−3.70	4.15	1.0660
Total	6.90	2.84	4.06	

Table 15

Cariño scaling

Period 1	Allocation (%)	Selection (%)
Equity	−0.7261	0.4586
Bond	−0.2420	2.5985
Total	−0.9681	3.0571

Period 2	Allocation (%)	Selection (%)
Equity	0.5579	−3.2385
Bond	1.0361	−0.8096
Total	1.5940	−4.0482

Period 3	Allocation (%)	Selection (%)
Equity	0.1279	1.2792
Bond	0.0320	2.9849
Total	0.1599	4.2641

Period 1–3	Allocation (%)	Selection (%)
Equity	−0.0403	−1.5007
Bond	0.8260	4.7738
Total	0.7858	3.2731
Excess	4.0589	
Residual	0.0000	

Table 16

	Portfolio (%)	Benchmark (%)	Arith. diff. (%)	Menchero (%)
Period 1	3.93	1.88	2.05	1.0443
Period 2	2.40	4.83	−2.43	1.0177
Period 3	0.45	−3.70	4.15	1.0568
Total	6.90	2.84	4.06	

benchmark return. With this assumption, the following simplified Frongello linking formula will provide roughly the same results.

$$F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \bar{R}_t \sum_{j=1}^{t-1} F_{jb}$$

The Frongello examples are given in Table 18.

Table 17

Menchero scaling

Period 1	**Allocation (%)**	**Selection (%)**
Equity	−0.7441	0.4700
Bond	−0.2480	2.6631
Total	−0.9921	3.1330

Period 2	**Allocation (%)**	**Selection (%)**
Equity	0.5610	−3.2566
Bond	1.0419	−0.8142
Total	1.6029	−4.0708

Period 3	**Allocation (%)**	**Selection (%)**
Equity	0.1268	1.2682
Bond	0.0317	2.9591
Total	0.1585	4.2274

Period 1–3	**Allocation (%)**	**Selection (%)**
Equity	−0.0563	−1.5185
Bond	0.8255	4.8080
Total	0.7693	3.2896
Excess	4.0589	
Residual	0.0000	

Table 18

	Modified Frongello		**Simplified Frongello**	
	Allocation (%)	**Selection (%)**	**Allocation (%)**	**Selection (%)**
Period 1				
Equity	−0.7125	0.4500	−0.7125	0.4500
Bond	−0.2375	2.5500	−0.2375	2.5500
Total	−0.9500	3.0000	−0.9500	3.0000
Period 2				
Equity	0.5415	−3.2765	0.5385	−3.3039
Bond	1.0449	−0.7311	1.0525	−0.7084
Total	1.5864	−4.0076	1.5910	−4.0123
Period 3				
Equity	0.1307	1.3252	0.1341	1.3826
Bond	0.0189	2.9554	0.0018	2.9116
Total	0.1496	4.2806	0.1359	4.2942
Period 1–3				
Equity	−0.0403	−1.5014	−0.0399	−1.4713
Bond	0.8262	4.7743	0.8167	4.7532
Total	0.7859	3.2729	0.7769	3.2820
Excess	4.0589		4.0589	
Residual	0.0000		0.0000	

Table 19 ■■■

	Portfolio (%)	Benchmark (%)	Arith. diff. (%)	Cariño coef. (%)	Menchero coef. (%)
Period 1	3.93	1.88	2.05	1.0190	1.0443
Period 2	2.40	4.83	−2.43	1.0120	1.0177
Period 3	0.45	−3.70	4.15	1.0660	1.0568
Total	6.90	2.84	4.06		
			St. Dev.	0.02936	0.02000

Review of passing algorithms

Although each of these algorithms will produce results that are not materially different from each other, it is important and interesting to get an understanding for why the answers from each of the preceding methods may differ. Philosophical differences, which flow through the mathematics, account for the small differences that occur. These topics will be covered next.

Return sensitivity

Should the level of total returns matter in scaling an attribute? In Cariño's scaling algorithm, periods of lower return experience a higher scaling coefficient and *vice versa*. This can be seen in the numerator of the Cariño scaling coefficient. Menchero, on the other hand, argues that there is no defensible basis for the scaling of an attribute to be based on the magnitude of the returns of the period. The Menchero scaling algorithm aims to produce a coefficient with minimal variance across periods. There is some evidence of this if the reader once again focuses on the period returns and coefficients produced. In Table 19 note that the Menchero scaling coefficients have a smaller standard deviation than the Cariño scaling coefficients. Menchero aims to minimise the difference in the scaling coefficients, while Cariño argues that the difference in the coefficients is a natural consequence of the compounding of attributes.

If one considers the intuition behind the Frongello method, the reader will find an argument and support for Cariño's rationale. The intuitive interpretation of the Frongello formula states the following.

1. Current attributes compound with past portfolio returns. Why? Because attributes are just a component of total return and

current returns compound over previously earned cumulative returns.

2. Current attributes compound into the future with the benchmark rate of return. Why? Well if no active bets are made in future periods (ie, passive indexing), then the relative return difference (attributes) will continue to grow at the benchmark rate of return.

Note that the total returns associated with the attributes of the period do not have an impact on the scaling of that attribute. A particular attribute will only grow by past portfolio return and future benchmark return, and not by the returns of the period in which the attribute occurs. Therefore, it makes sense that if an attribute occurs in a period with relatively high total returns, that attribute will not be scaled heavily because those returns will not have an impact. Similarly, an attribute that occurs in a period of relatively low returns will be scaled by the relatively higher returns occurring in other periods, resulting in a high level of scaling. In conclusion, the Frongello and Cariño methods disagree with Menchero's argument for similar scaling regardless of the relative return level.

Acausality

Mirabelli (2000–2001) attempted to resolve an interesting condition that the Cariño and Menchero coefficients depend upon. Andre Mirabelli noticed that the coefficient algorithms required information about the returns of the entire cumulative period in order to calculate the coefficients of any individual period. The reader will notice that the total returns of the portfolio and benchmark for the entire cumulative period are necessary inputs into the coefficient formulas. For example, Mirabelli noticed that to scale the results for January, February and March to be used in the eventual one-year analysis, the analyst would need information about portfolio and benchmark returns that have not occurred yet. Mirabelli offered the argument that a linking algorithm should be non-acausal or in other words the linking methodology should not be dependent on future events when scaling single-period results. His solution was a recursive algorithm that would explain the outperformance gained in each period only after that amount could be measured. Mirabelli identified an interesting challenge to the coefficient methods and made significant contributions toward the development of an accurate recursive solution.

Inspired by Mirabelli, the Frongello methods were engineered to be non-acausal. All the information required to scale the attribute of the current period is available in the current period. The Frongello methods are not dependent on future returns.

Order dependence

If we were to reverse the order of periods in our study, all the methods would return the same answer except for the Frongello methods. The modified Frongello method has a very small almost undetectable difference with reversed periods, the difference is immaterial and for most situations the modified Frongello method can be thought of as order-independent.[9] The simplified Frongello algorithm is order dependent. The differences are noticeable but arguably materially insignificant. Most performance specialists sacrifice the order independence of the modified Frongello for the simplicity of the simplified Frongello algorithm. Again, the order dependent simplified Frongello method will deviate from the other methods most when the returns do not track the benchmark over time. For example, with absolute attribution[10] the simplified Frongello method is not an appropriate alternative.

Sincerity

For lack of a better term, sincerity explains the mechanical differences between the single-period coefficient scaling of Cariño and Menchero and the recursive scaling of Frongello. With coefficient scaling, coefficients are used to increase the impact of known attributes by use of a scalar to capture the effect of compounding. There are pros and cons to this approach. The negative aspect of coefficient scaling is that single-period scaled results will have to be adjusted every time a new period is added to the analysis because the total return inputs into the coefficient have changed resulting in new scaling coefficients. Most portfolio managers dislike seeing their history restated. Also, the contribution to excess return of a particular period is not attributed in the period in

Table 20

	Port. (%)	Bench. (%)	Diff. (%)	Alloc. (%)	Select. (%)
Frongello					
Period 1	21.00	11.00	10.00	**6.00**	**4.00**
Period 2	14.00	9.00	5.00	**2.96**	**3.99**
Total	37.94	20.99	16.95	8.96	7.99
Period 1	21.00	11.00	10.00	**6.00**	**4.00**
Period 2	14.00	9.00	5.00	**2.96**	**3.99**
Period 3	20.00	12.00	8.00	2.45	10.61
Total	65.53	35.51	30.02	11.41	18.60
Menchero					
Period 1	21.00	11.00	10.00	**6.77**	**4.51**
Period 2	14.00	9.00	5.00	**2.27**	**3.40**
Total	37.94	20.99	16.95	9.04	7.91
Period 1	21.00	11.00	10.00	**7.82**	**5.22**
Period 2	14.00	9.00	5.00	**2.61**	**3.92**
Period 3	20.00	12.00	8.00	1.31	9.14
Total	65.53	35.51	30.02	11.74	18.28

which it occurs. The examples shown in Table 20 will make this clearer.

Here we compare the simplified Frongello recursive method with the Menchero coefficient method. Note that the Menchero periods 1 and 2 results change when period 3 is added to the analysis. The Frongello method maintains the prior results. Furthermore, note that the Frongello method explains the cumulative outperformance of 10% in period 1 by attributes that add to 10% in period 1. During period 2, the cumulative outperformance increases to 16.95%. That increase of 6.95% is explained in period 2. This is not the case with the Menchero presentation.

The presentation of the results from the Frongello method is not without criticism. Consider a similar example (see Table 21), where the manager indexes in periods 2 and 3.

Note here that due to indexing, there are no active attributes in periods 2 and 3. The Frongello method, however, reports an allocation and selection effect in periods 2 and 3. Some argue that there should not be any linking adjustment in these periods because no single-period effect has occurred. Others are comfortable recognising this adjustment as an echo of past effects

Table 21

	Port. (%)	Bench. (%)	Diff. (%)	Alloc. (%)	Select. (%)
Frongello					
Period 1	21.00	11.00	10.00	6.00	4.00
Period 2	9.00	9.00	0.00	**0.54**	**0.36**
Total	31.89	20.99	10.90	6.54	4.36
Period 1	21.00	11.00	10.00	6.00	4.00
Period 2	9.00	9.00	0.00	**0.54**	**0.36**
Period 3	12.00	12.00	0.00	**0.78**	**0.52**
Total	47.72	35.51	12.21	7.32	4.88
Menchero					
Period 1	21.00	11.00	10.00	6.54	4.36
Period 2	9.00	9.00	0.00	**0.00**	**0.00**
Total	31.89	20.99	10.90	6.54	4.36
Period 1	21.00	11.00	10.00	7.32	4.88
Period 2	9.00	9.00	0.00	**0.00**	**0.00**
Period 3	12.00	12.00	0.00	**0.00**	**0.00**
Total	47.72	35.51	12.21	7.32	4.88

compounding at some passive rate (the benchmark rate in the simplified Frongello method). Bear in mind that this distinction is only reflected in the single-period support, cumulative results will be immaterially different between the methods. Most importantly, note that history must be restated or an echo effect must be present. This is really a matter of preference for the analyst and their clients.

Linking summary

At this point we have reviewed the popular linking methods and provided descriptions and examples for the small differences one might see when reviewing methods. One important point that should resound with the reader is that although the mathematics can differ greatly from one method to another, the story told by the resulting attribution is the same regardless of the method. Table 22 illustrates that even when you consider geometric and arithmetic results together, all of the results present roughly the same intuitive story and it can easily be argued that the differences among the presentations are immaterial.

Table 22

	Allocation (%)	Selection (%)	Total (%)
Geometric			
BKT	0.71	3.21	3.92
BKT exponential	0.75	3.17	3.92
Pure geometric	0.77	3.22	3.99
Adj. pure – Cariño	0.83	3.09	3.92
Adj. pure – Menchero	0.76	3.16	3.92
Arithmetic			
Cariño scaling	0.79	3.27	4.06
Menchero scaling	0.77	3.29	4.06
Modified Frongello	0.79	3.27	4.06
Simplified Frongello	0.78	3.28	4.06

FUTURE CHALLENGES

Daily attribution and accounting for risks

Some argue, "Why go daily? We do not make investment management decisions everyday in our portfolios, so why measure performance and attribution daily? Should we not just worry about matching the performance and attribution process to our investment management process?" This is actually a very popular mentality. However, something is missed in these statements. Performance measurement and attribution should not just measure the influence of active decisions. The larger goal of performance measurement and attribution is to measure the relative impact of risk factor mismatches *versus* an index over a market cycle. Performance attribution answers the question "Why did my portfolio perform differently from the benchmark?" In an ideal world, we would like to think that all risk factor mismatches *versus* the benchmark are present due to diligent and intentional management by the portfolio manager. However, in the real world, risk factor mismatches may occur unintentionally due to neglect, accidents or the simple evolution of the risk profile of securities.[11] Performance attribution should seek to explain performance due to all of these risk factor mismatches and only then should the secondary concern be to identify the source of that mismatch. Daily attribution is the best method for monitoring the status and consequence of risk factor mismatches. With the linking tools presented in this chapter, the performance analysis is empowered to increase measurement frequency, resulting in more accurate and timely analyses of managed portfolios.

1 Nicely summarised by Spaulding (2003).
2 However, there is some difficulty in arriving at the compoundable geometric attributes.
3 For a nice review of the evolution that led to this model see Spaulding (2003).
4 A variable present under the sum or product symbol indicates that all occurrences of that variable are to be included.
5 Recall that the BF attributes are additive and originally sum to the attribute totals. These totals then sum to the arithmetic excess return.
6 This debatable criticism was offered by Menchero (2005).
7 Spaulding and Bacon found that the geometric approach is more popular in Europe.
8 Again, see Spaulding (2003) for a nice summary of fixed income and currency decompositions.
9 To many decimal points in almost all occasions the Cariño and modified Frongello methods provide identical results.
10 Absolute return is performance not relative to benchmark. The reader can think of absolute attribution as relative attribution to a benchmark return of zero.
11 A good example of this is fixed income securities. Durations change everyday. Even in a seldom-traded portfolio, a duration-matched position on one day will most certainly not be as closely matched on the following day.

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