Linking Single Period Attribution Results

While methods of single period attribution abound, accurate linking methodologies are limited. The author presents a linking methodology that seeks to retain explanatory power while presenting multiple period attribution results by means of a method friendly to an audience without advanced degrees in mathematics.

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ATTRIBUTION

Attribution analysis is the study of explaining a portfolio's performance relative to a benchmark, over a given time frame, among a set of predetermined effects. Despite the abundance of single period attribution methodologies[1], there continues to be no clear industry standard for linking these single period results. This lack is clearly due to the complexity of compounding attributes over multiple periods and no industry wide consensus on desirable linking methodology characteristics. In the following paragraphs I will put forth some preferred characteristics of single and multiple period attribution methodologies. I will also address the linking challenge and propose an algebraic linking solution, Frongello linking.

SINGLE PERIOD

Attribution analysis informs those concerned with how active management performed relative to a benchmark over a reporting period. From a top-down prospective, those concerned will want to know how the manager performed from an allocation standpoint. By bucketing the portfolio by duration, quality, sector, industry, P/E Ratio, etc., the portfolio manager can discover if their active weighting versus the benchmark contributed to over/underperformance due to allocation. From the bottom-up perspective, those concerned will want to determine the portfolio manager's ability to pick outperforming securities. Once the portfolio is bucketed, the return of the manager's buckets can be compared to similar buckets in the benchmark to indicate any contribution to over/underperformance due to selection. Allocation and selection effects are by far the most common, and more importantly, most intuitive effects in attribution schemes today.[2]

Single period methodologies today not only differ in regards to which attributes to present but also in regards to *how* these attributes are presented. Attribution effects can be presented "Geometrically,"[3] where the attributes are typically represented by a ratio that is multiplicative across periods to arrive at a cumulative ratio. Unfortunately, the end results of geometric methods are somewhat unintuitive in interpretation. More often however, the returns are presented in an additive fashion. The appeal of the additive presentation stems from the method's conveyance of information in a simple, straightforward and intuitive manner. Although quantitative mathematicians generate attribution statistics, focus should be given towards the audience of those results. This audience may include individuals without highly mathematical backgrounds, such as: corporate executives, consultants, portfolio managers, relationship managers and clients. Throughout this paper, the attribution scheme used is illustrated in Figure 1. This single period attribution scheme's benefits include:

1. additive results, and

2. easily interpreted effects.

With numerous methods of single period attribution schemes available to the analyst, the scheme presented here makes no sacrifice in explanatory power while presenting single period attribution results in a method friendly to an audience without advanced degrees in mathematics. In summary, the additive approach is arguably the most appropriate.

Figure 1
Attribution Effects.

$$\sum_i W_{it} R_{it} = R_t$$

$$\sum_i W_{it} = 1$$

$$R_t - \overline{R}_t = S_t + A_t$$

$$S_t = \sum_i [W_{it}(R_{it} - \overline{R}_{it})]$$

$$A_t = \sum_i [(W_{it} - \overline{W}_{it})(\overline{R}_{it} - \overline{R}_t)]$$

R_t = Return of portfolio during period t

$\overline{R}_t$ = Return of benchmark during period t

S_t = Selection effect of all sectors i in period t

A_t = Allocation effect of all sectors i in period t

R_{it} = Return of sector i in portfolio during period t

$\overline{R}_{it}$ = Return of sector i in benchmark during period t

W_{it} = Weight of sector i in portfolio during period t

$\overline{W}_{it}$ = Weight of sector i in benchmark during period t

THE LINKING CHALLENGE

While this methodology illustrates the results of allocation and selection effects over a given period, often the audience is interested in analyzing active management decision results over multiple periods. Unfortunately, although returns are easily compounded from period to period, attribution effects are much more complicated to aggregate over multiple periods. David Cariño (1999) (pp. 56-57) beautifully illustrated the linking problem. He noted that while the compound return of a portfolio equals,

$$R = (1 + R_1)(1 + R_2) \ldots (1 + R_n) - 1$$

and the compound return of the benchmark equals,

$$\overline{R} = (1 + \overline{R}_1)(1 + \overline{R}_2) \ldots (1 + \overline{R}_n) - 1$$

the sum of return differences (or alternatively, sum of relative attribution effects) does not equal the difference in compounded total return.

$$R - \overline{R} \neq (R_1 - \overline{R}_1) + (R_2 - \overline{R}_2) \ldots (R_n - \overline{R}_n)$$

David Cariño also noted that compounding the differences in returns (or alternatively, summing or compounding the compound of relative attribution effects) will not work either.

$$R - \overline{R} \neq (1 + R_1 - \overline{R}_1)(1 + R_2 - \overline{R}_2) \ldots (1 + R_n - \overline{R}_n) - 1$$

The challenge remains to relate single period attribution results to cumulative over/underperformance. In the following section, I will put forth a list of standards by which to judge potential solutions to this challenge.

STANDARDS OF JUDGEMENT

Algorithms[4] have been developed that attempt to address this compounding challenge. However, only a few remain after screening the population by the following three necessary and/or desirable characteristics proposed by David Cariño in the summer of 1999.

- *Generality* – (Cariño 1999, p. 6)The linking methodology is independent of the single period attribution scheme used. The linking methodology should work regardless of security bucketing decisions, currency effects, interest in interaction, or the mathematics used to attain these results. The single period attributes in question must add to the relative difference in return.

- *Familiarity* – (Cariño 1999, p. 6) The interpretation of single period results should not differ from the interpretation of multiple period results.

- *No Residuals/Distortion* – (Cariño 1999, p. 6) The linking methodology should attribute the whole, and only the whole, of the relative over/underperformance.

In addition to these characteristics put forth by David Cariño, I believe an optimal attribution linking methodology should also satisfy the three characteristics which I propose here:

- *Sincerity* – The linking methodology should put forth results that are as close to reality as possible. The model should be devoid of any mathematical fudging used in order to satisfy any of the desirable characteristics.

- *Intuitive* – The linking methodology mathematics should preferably involve mathematics that can be accepted and understood by an audience without advanced degrees in mathematics. The audience using the attribution results should have a comfortable understanding of the linking mathematics and why they work.

- *Order Dependence* – While the order of periods has no bearing on the resulting cumulative total return, the order of periods does have a bearing on the cumulative attribution results. The methodology should not ignore the importance of order dependence and it's effect on accurate cumulative results.

THE CHALLENGE ILLUSTRATED

I plan to propose a method that satisfies these characteristics, the Frongello Linking Methodology, and compare this method to two other respectable linking methodologies: the Cariño and the Menchero methodologies. To help illustrate the linking problem, Figure 2 analy-

Figure 2
Analysis of the Attribution of a Stock and Bond Portfolio Over Three Identical Periods.

	Portfolio		Benchmark		Attribution	
Periods 1,2,3	Weight	Return	Weight	Return	Allocation	Selection
Stock	80%	6.00%	60%	5.00%	0.24%	0.80%
Bond	20%	3.00%	40%	2.00%	0.36%	0.20%
Total	100%	5.40%	100%	3.80%	0.60%	1.00%

	Return			Allocation		Selection		
	Portfolio	Benchmark	Difference	Stock	Bond	Stock	Bond	Residual
Period 1	5.4000%	3.8000%	1.6000%	0.2400%	0.3600%	0.8000%	0.2000%	0.0000%
Period 2	5.4000%	3.8000%	1.6000%	0.2400%	0.3600%	0.8000%	0.2000%	0.0000%
Period 3	5.4000%	3.8000%	1.6000%	0.2400%	0.3600%	0.8000%	0.2000%	0.0000%
Cumulative	17.0905%	11.8387%	5.2519%	0.7200%	1.0800%	2.4000%	0.6000%	0.4519%
	17.0905%	11.8387%	5.2519%	*0.7217%*	*1.0839%*	*2.4193%*	*0.6012%*	0.4258%

ses the attribution of a stock and bond portfolio over three identical periods.

You'll notice that the sum of each attribute (underlined) and the product of each attribute (*italics*), for each sector, for each period, leaves a substantial unexplained residual. Defining the variable G_{itb} as the effect due to sector "i" in time period "t" for attribute "b" we can say that the sum of all variables G_{itb} does not equal the difference in cumulative return.

$$\sum_i \sum_t \sum_b G_{itb} \neq R - \overline{R} .$$

To benefit the multi-period analysis, the proper scaling must be applied to the variables G_{itb} so that the sum of the scaled G_{itb} (scaled G_{itb} denoted by F_{itb}) equals the difference in cumulative return.

$$\sum_i \sum_t \sum_b F_{itb} = R - \overline{R} .$$

THE FRONGELLO LINKING ALGORITHM

The Frongello scaling algorithm follows:

$$F_{itb} = G_{itb}\left(\prod_{j=1}^{t-1}(1+R_j)\right) + \overline{R}_t\left(\sum_{j=1}^{t-1} F_{ijb}\right) .$$

We can easily decompose the Frongello algorithm into its parts. Throughout this reasoning, remember that the portfolio return equals the sum of the relative attributes plus the benchmark or "passive" return. The first part of the equation simply represents an attribution effect (of sector "i", in time "t" and due to effect "b") multiplied by the cumulative return of the portfolio through the prior period. This is done for a very simple and intuitive reason. The single period return due to this attribute compounds over any cumulative portfolio performance achieved before the attribute occurs. The second part of the equation recognizes that the sum of the adjusted attributes through the prior period increases by the current period benchmark return, or "passive" return of the portfolio. This is done because the sum of all prior attributes will compound with the current benchmark or "passive" return. Current attributes are treated separately from prior attributes so not to inflate current single period attribution effects with prior single period attribution effects. The decomposition simply scales the current attribute's effect and the attribute's prior effects separately. The current attribute is scaled by the cumulative return of the portfolio through the prior period and the sum of all prior scaled attributes is further scaled by the current benchmark or "passive" return. This mathematical line ensures the proper treatment of order dependence[5]. The results for the Frongello solution follow in Figure 3 (*see page* 14).

In Figure 3, we've used the Frongello linking methodology to transform G_{itb} to F_{itb}, which enables us to sum the single period attributes to arrive at the cumulative results. These cumulative results explain the exact difference in cumulative returns. In addition, we have satisfied our earlier discussed desirable characteristics, which include:

1. *Generality* – Although we used a very simple example, the method used is the same regardless of the single period scheme or attributes calculated.

2. *Familiarity* – The multiple period results are interpreted in the same fashion as our single period results.

3. *No Residuals/Distortions* – Our multiple period analysis explains exactly the total difference in cumulative return.

4. *Sincerity* – Although the inputs of attribution analysis are rough approximations of reality (because weights and returns are approximated after accounting for flows) we must accept these unavoidable single period approximations. Applying some simple high school algebra and return mathematics,[5] our multiple period results are as accurate to reality as the approximate single period inputs allow. However, the Frongello linking alone is a mathematical depiction of reality.

5. *Intuitive* – The scaling rational is straightforward, logical, and most importantly simple to understand.

6. *Order Dependence* – The Frongello method appropriately addresses the importance of order dependence by inflating attributes according to their order of occurrence in the cumulative period.

THE CARIÑO LINKING ALGORITHM

David Cariño proposed another very elegant additive methodology during the Summer of 1999 (p. 8). He scaled attributes in the following manner:

$$F_{itb} = G_{itb}(K_t/K)$$
$$K_t = [\ln(1+R_t) - \ln(1+\overline{R}_t)]/(R_t - \overline{R}_t)$$
$$K = [\ln(1+R) - \ln(1+\overline{R})]/(R - \overline{R})$$

The Cariño methodology scales returns by mathematically recognizing the relationship between nominal returns and the log of these nominal returns. Although appealing because the results introduce no residual, this is achieved by systematically distributing the method's inevitable residuals among the attributes. Cariño notes in his paper that in the single period, a residual results from the disjoint between the difference in returns (denominator in K_t) and the difference in the log of the returns (numerator in K_t). He states, "the residual ... is distributed throughout the table by multiplying the additive effects by the factor K_t" (p. 10). Later, to arrive at the final cumulative results, he introduces another adjustment to reconcile the disjoint between the difference in cumulative returns (denominator in K) and the difference in the log of the cumulative returns (numerator in K). He notes, "To calculate the additive effects, the residual ... was distributed proportionately among the elements by the factor K" (p. 11). Not only is there significant evidence of mathematical residual burying, but I also feel that an intuitive interpretation of this mathematical line is incredibly difficult. This evidence indicates a violation of sincerity and intuitiveness. Furthermore, the Cariño method makes no attempt to recognize the importance of order dependence. A period's attribution is treated the same regardless of it's order of occurrence during the cumulative period. Jose Menchero also notes in his Fall 2000 paper that although the Cariño approach's scaling coefficients produce no residual, "the logarithmic coefficients tend to overweight periods with lower-than-average returns, and to underweight those with higher-than-average returns" (p. 39). This can be seen in K_t. This further casts doubts on the methodology's sincerity characteristic. Menchero attempts to address this apparent bias in his Fall 2000 methodology. Although the Cariño methodology satisfies the characteristics of generality, familiarity, and no residuals/distortion, a

Figure 3
Frongello Solution Results.

	Portfolio		Benchmark		Attribution	
Periods 1,2,3	Weight	Return	Weight	Return	Allocation	Selection
Stock	80%	6.00%	60%	5.00%	0.24%	0.80%
Bond	20%	3.00%	40%	2.00%	0.36%	0.20%
Total	100%	5.40%	100%	3.80%	0.60%	1.00%

	Return			Allocation		Selection		
	Portfolio	Benchmark	Difference	Stock	Bond	Stock	Bond	Residual
Period 1	5.4000%	3.8000%	1.6000%	0.2400%	0.3600%	0.8000%	0.2000%	
Period 2	5.4000%	3.8000%	1.6000%	0.2621%	0.3931%	0.8736%	0.2184%	
Period 3	5.4000%	3.8000%	1.6000%	0.2857%	0.4285%	0.9523%	0.2381%	
Cumulative	17.0905%	11.8387%	5.2519%	*0.7878%*	*1.1817%*	*2.6259%*	*0.6565%*	0.0000%

Figure 4
Cariño Solution Results.

Periods 1,2,3	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return	Attribution Allocation	Attribution Selection
Stock	80%	6.00%	60%	5.00%	0.24%	0.80%
Bond	20%	3.00%	40%	2.00%	0.36%	0.20%
Total	100%	5.40%	100%	3.80%	0.60%	1.00%

	Return			Allocation		Selection		
	Portfolio	Benchmark	Difference	Stock	Bond	Stock	Bond	Residual
Period 1	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	
Period 2	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	
Period 3	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	
Cumulative	17.0905%	11.8387%	5.2519%	*0.7878%*	*1.1817%*	*2.6259%*	*0.6565%*	0.0000%

critical eye must be cast on the method's sincerity, intuitiveness, and order dependence. The Cariño solution follows in Figure 4.

THE MENCHERO LINKING ALGORITHM

The last additive linking methodology worth mentioning is the Menchero methodology proposed by Jose Menchero during the Fall of 2000. In contrast to the Cariño methodology, Menchero's scaling consciously attempts to weight each period as evenly as possible. He scales attributes as follows:

$$F_{itb} = G_{itb}(A + \alpha_t).$$

Where,

T = Number of periods in a multiple period

$A = (1/T)[(R - \overline{R})/((1+R)^{1/T} - (1+\overline{R})^{1/T})], (R \neq \overline{R})$

$A = (1+R)^{(T-1)/T}, (R = \overline{R})$

$\alpha_t = [(R - \overline{R} - A\sum_{j=1}^{T}(R_j - \overline{R}_j))/\sum_{j=1}^{T}(R_j - \overline{R}_j)^2](R_t - \overline{R}_t)$

In the first page of Jose Menchero's Fall 2000 paper he notes two critical points to the methodology. He notes, "the first point is to recognize that geometric compounding leads to a geometric scaling law, which relates the single-period excess returns to the linked excess returns" (p. 1). The variable A in the above equation describes this portion. He acknowledges a resulting small residual. He notes, "the second point concerns the optimal distribution of this small residual among the different periods to produce a residual-free linking algorithm." (p. 1). This mathematical fudging is represented by variable α_t in the prior equation. Again we see a linking methodology that satisfies the characteristics of generality, familiarity and no residual/distortion. However, I believe the characteristics of sincerity, intuitiveness, and order dependence are violated. First, one without an advanced degree in mathematics would have difficulty understanding the intuitive rational behind Menchero's geometric scaling coefficient and corrective term. Second, the corrective term in itself challenges the sincerity of the mathematics. Sacrificing sincerity in order to accomplish a model with no residual is not an optimal solution. This linking methodology introduces

Figure 5
Menchero Solution Results.

Periods 1,2,3	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return	Attribution Allocation	Attribution Selection
Stock	80%	6.00%	60%	5.00%	0.24%	0.80%
Bond	20%	3.00%	40%	2.00%	0.36%	0.20%
Total	100%	5.40%	100%	3.80%	0.60%	1.00%

	Return			Allocation		Selection		
	Portfolio	Benchmark	Difference	Stock	Bond	Stock	Bond	Residual
Period 1	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	0.0000%
Period 2	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	0.0000%
Period 3	5.4000%	3.8000%	1.6000%	0.2626%	0.3939%	0.8753%	0.2188%	0.0000%
Cumulative	17.0905%	11.8387%	5.2519%	*0.7878%*	*1.1817%*	*2.6259%*	*0.6565%*	0.0000%

approximations in addition to the approximations of the single period inputs. Therefore, the methodology's use of mathematical fudging is not sincere to an accurate depiction of reality. Lastly, the Menchero algorithm also fails to recognize the effects of order dependence and subsequently makes the same mistake as Cariño's method. The Menchero solution follows in Figure 5.

SUBTLE DIFFERENCES

You may have noticed that the three methods discussed in this paper yield the same cumulative results. This happens because although the scaling treatments differ across methodologies, when periods in question are identical in return and attribution, the methodologies capture the same cumulative scaling. First, because the periods are identical, it is impossible to challenge and distort results by ignoring order dependence. Second, because the Cariño and Menchero methods differ in regard to how they weight single period attribution results by the single period returns, periods with identical returns will be weighted identically between these methods. In the special case of identical periods, the three methods discussed here will have identical cumulative attribution, although only the Cariño and Menchero methods will have identical single period scaling. The Frongello method's single period scaling will differ in this special case because it is the only method to acknowledge order dependence in it's single period scaling. However, the subtle differences become apparent when periods with unique results come under question. When looking at unique periods, each method will produce differing single period scaling and cumulative attribution. A comparison of the three methodologies under three unique periods follows in Figure 6 (*see page* 17).

While Cariño's scaling approximation tends to overweight periods of below average returns, Menchero's scaling approximation attempts to weight periods as evenly as possible. Unfortunately, these two methods both fail to take into account order dependence, a negligence that can introduce significant errors. The Frongello method rather avoids scaling approximation all together by letting the natural scaling run its course over the periods in question. All scaling done is reflective of fundamental return mathematics based on the results of a single period scheme. The single period results define the proper scal-

Figure 6
Comparison of the Frongello, Cariño, and Menchero Solutions Under Three Unique Periods.

Period 1	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	60.00%	14.00%	50.00%	11.00%
Bond	40.00%	10.00%	50.00%	12.00%
Total	100.00%	12.40%	100.00%	11.50%

	Allocation	Selection	Total
Stock	–0.05%	1.80%	1.75%
Bond	–0.05%	–0.80%	–0.85%
Total	–0.10%	1.00%	0.90%

Period 2	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	70.00%	6.00%	40.00%	7.00%
Bond	30.00%	3.00%	60.00%	4.00%
Total	100.00%	5.10%	100.00%	5.20%

	Allocation	Selection	Total
Stock	0.54%	–0.70%	–0.16%
Bond	0.36%	–0.30%	0.06%
Total	0.90%	–1.00%	–0.10%

Period 3	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	30.00%	10.00%	60.00%	9.00%
Bond	70.00%	8.00%	40.00%	5.00%
Total	100.00%	8.60%	100.00%	7.40%

	Allocation	Selection	Total
Stock	–0.48%	0.30%	–0.18%
Bond	–0.72%	2.10%	1.38%
Total	–1.20%	2.40%	1.20%

	Portfolio	Benchmark	Diff
Frongello	28.2918%	25.9781%	2.3137%
Cariño	28.2918%	25.9781%	2.3137%
Menchero	28.2918%	25.9781%	2.3137%

Allocation Stock	Allocation Bond	Allocation Total	Selection Stock	Selection Bond	Selection Total	Residual
0.0283%	–0.4725%	–0.4441%	1.5431%	1.2147%	2.7578%	0.0000
0.0311%	–0.4691%	–0.4380%	1.5509%	1.2008%	2.7517%	0.0000
0.0217%	–0.4672%	–0.4455%	1.6127%	1.1465%	2.7592%	0.0000

ing from period to period and properly acknowledge the order dependence in the final linked results. The return mathematics used are elementary and the Frongello method can be proved with high school level algebra.

The slight differences in results, among the methods, arise due to scaling differences when periods in question differ in return, attribution and the historical order of these statistics. The differences in the cumulative results, among the methods, will increase as:

1. The level of returns increases,
2. The variation in the single period returns and attribution increases, and/or
3. The number of single periods in the cumulative period increases.

The value added by the Frongello linking methodology becomes apparent under these conditions.

Figure 7
Cumulative Results During a Cumulative Period.

Period 1	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	70.00%	45.00%	50.00%	-25.00%
Bond	30.00%	5.00%	50.00%	-5.00%
Total	100.00%	33.00%	100.00%	-15.00%

	Allocation	Selection	Total
Stock	–2.00%	49.00%	47.00%
Bond	–2.00%	3.00%	1.00%
Total	–4.00%	52.00%	48.00%

Period 2	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	15.00%	40.00%	10.00%	10.00%
Bond	85.00%	40.00%	90.00%	20.00%
Total	100.00%	40.00%	100.00%	19.00%

	Allocation	Selection	Total
Stock	–0.45%	4.50%	4.05%
Bond	–0.05%	17.00%	16.95%
Total	–0.50%	21.50%	21.00%

Period 3	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	95.00%	5.00%	10.00%	45.00%
Bond	5.00%	45.00%	90.00%	5.00%
Total	100.00%	7.00%	100.00%	9.00%

	Allocation	Selection	Total
Stock	30.60%	–38.00%	–7.40%
Bond	3.40%	2.00%	5.40%
Total	34.00%	–36.00%	–2.00%

	Portfolio	Benchmark	Diff
Frongello	99.2340%	10.2535%	88.9805%
Cariño	99.2340%	10.2535%	88.9805%
Menchero	99.2340%	10.2535%	88.9805%

Allocation Stock	Allocation Bond	Allocation Total	Selection Stock	Selection Bond	Selection Total	Residual
53.7306%	3.6641%	57.3948%	–0.6745%	32.2602%	31.5858%	0.0000
39.2804%	1.8710%	41.1514%	21.0517%	26.7774%	47.8291%	0.0000
37.0233%	1.7493%	38.7726%	21.1014%	29.1064%	50.2079%	0.0000

OBVIOUS DIFFERENCES

In Figure 7 we look at the cumulative results during a cumulative period that experiences large returns and return variances.

You'll notice that while the Menchero and Cariño methods produce results that are relatively similar, the Frongello method produces quite a different result. Total outperformance due to stock selection is roughly 20% larger in the Menchero and Cariño methods. This difference is largely due to the Frongello method's proper treatment of order dependence. The attribution in the second period is dependent on the attribution results of the first period, the attribution results in the third period are dependent on the attribution results in the first and second period, and so on and so forth. The Menchero and Cariño methods pay no attention to order dependence. In Figure 8 (*see page* 19), it is clear that even when the periods are arranged in reverse order, the

Figure 8
Cumulative Results Reverse Arrangement.

Period 3	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	95.00%	5.00%	10.00%	45.00%
Bond	5.00%	45.00%	90.00%	5.00%
Total	100.00%	7.00%	100.00%	9.00%

	Allocation	Selection	Total
Stock	30.60%	–38.00%	–7.40%
Bond	3.40%	2.00%	5.40%
Total	34.00%	–36.00%	–2.00%

Period 2	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	15.00%	40.00%	10.00%	10.00%
Bond	85.00%	40.00%	90.00%	20.00%
Total	100.00%	40.00%	100.00%	19.00%

	Allocation	Selection	Total
Stock	–0.45%	4.50%	4.05%
Bond	–0.05%	17.00%	16.95%
Total	–0.50%	21.50%	21.00%

Period 1	Portfolio Weight	Portfolio Return	Benchmark Weight	Benchmark Return
Stock	70.00%	45.00%	50.00%	-25.00%
Bond	30.00%	5.00%	50.00%	-5.00%
Total	100.00%	33.00%	100.00%	-15.00%

	Allocation	Selection	Total
Stock	–2.00%	49.00%	47.00%
Bond	–2.00%	3.00%	1.00%
Total	–4.00%	52.00%	48.00%

	Portfolio	Benchmark	Diff
Frongello	99.2340%	10.2535%	88.9805%
Cariño	99.2340%	10.2535%	88.9805%
Menchero	99.2340%	10.2535%	88.9805%

Allocation Stock	Allocation Bond	Allocation Total	Selection Stock	Selection Bond	Selection Total	Residual
27.5466%	0.3976%	27.9443%	39.0578%	21.9785%	61.0363%	0.0000
39.2804%	1.8710%	41.1514%	21.0517%	26.7774%	47.8291%	0.0000
37.0233%	1.7493%	38.7726%	21.1014%	29.1064%	50.2079%	0.0000

Menchero and Cariño methods produce results identical to Figure 7 (*see page* 18). However, the Frongello method accurately reflects the changed cumulative attribution results by acknowledging order dependence.

CONCLUSION

The linking methodologies discussed in this paper address the linking of single period additive attribution results. Additive results are generally favored due to their intuitive appeal. The three methods reviewed produce cumulative additive attribution that adheres to the characteristics defined by Cariño in the summer of 1999. These characteristics include generality, familiarity, and no residuals/distortions. However, evidence indicates that the Cariño and Menchero methods fail to satisfy the newly introduced characteristics of sincerity, order dependence, and intuitiveness. While the Frongello method provides similar results, I believe it is superior in that it:

1. provides a more accurate description of reality devoid of any mathematical rhetoric or "fudging" (Sincerity);

2. recognizes the impact of the historical order of periods on attribution effects (Order Dependence); and

3. Is more intuitive, straightforward, and appropriate for its audience (Intuitive).

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ENDNOTES

[1] Fama (1972),Brinson and Fachler (1985), Dietz, Fogler, Hardy (1980), Rudd and Clasing (1982), Fong, Pearson, and Vasicek (1983), Allen (1991), Ankrim (1992), Ankrim and Hensel (1994), Karnovsky and Singer (1994), Burnie, Knowles and Teder (1998), Singer, Gonzalo, and Lederman (1998).

[2] A few analysts prefer to calculate security selection using the benchmark weight of the sector instead of the portfolio weight. When attribution is calculated in this fashion an additional effect called interaction is introduced, $\text{Interaction} = (W_{it} - \overline{W}_{it})(R_{it} - \overline{R}_{it})$.

[3] Burnie, Knowles and Teder(1998), Menchero(2001)

[4] Menchero (2000), Kirievsky and Kirievsky (2000), Cariño (1999), Frongello (2002), Singer (1998).

[5]

Periods 1	Benchmark Weight	Benchmark Return	Portfolio Weight	Portfolio Return	Attribution Allocation	Attribution Selection
Stock	60.00%	5.00%	70.00%	7.00%	0.12%	1.40%
Bond	40.00%	2.00%	30.00%	3.00%	0.18%	0.30%
Total		3.80%		5.80%	0.30%	1.70%

Periods 2	Benchmark Weight	Benchmark Return	Portfolio Weight	Portfolio Return	Attribution Allocation	Attribution Selection
Stock	30.00%	5.00%	60.00%	6.00%	0.42%	0.60%
Bond	70.00%	3.00%	40.00%	5.00%	0.18%	0.80%
Total		3.60%		5.60%	0.60%	1.40%
Cumulative Return:		7.54%		11.72%		3.10%

	Benchmark		Portfolio		Difference	Allocation		Selection		Residual
Start	$ 1,000.00	Passive	$ 1,000.00	Passive		Stock	Bond	Stock	Bond	
After Period 1	$ 1,038.00	$ 38.00	$ 1,058.00	$ 38.00		$ 1.20	$ 1.80	$ 14.00	$ 3.00	
After Period 2	$ 1,075.37	$ 37.37	$ 1,117.25	$ 38.09		$ 4.44	$ 1.90	$ 6.35	$ 8.46	
Total	$ 75.37		$ 117.25		$ 41.88	$ 5.64	$ 3.70	$ 20.35	$11.46	$ 0.72

With Adjustments to Period 2	Stock	Bond	Stock	Bond
After Period 1	$ 1.20	$ 1.80	$ 14.00	$ 3.00
After Period 2	$ 4.49	$ 1.97	$ 6.85	$ 8.57
Total	$ 5.69	$ 3.77	$ 20.85	$ 11.57

In the example above, in each period the starting value is multiplied by the passive return and each attribute in order to find the dollar return due to each. Unfortunately however, the sum of the attributes' dollar returns ($5.64 + $3.70 + $20.35 + 11.46 = $41.16) does not add to the total dollar difference in return ($117.25 – $75.37 = $41.88). Why? Because our outperformance in period 1 allowed us to have a higher base to earn the passive return in period 2. $38.09 – $37.37 = $.72, which is the exact amount we are off. Obviously we owe this $.72 to our attributes in period one. If we simply take the dollar return of each attribute in period 1 and multiply them by the passive return in period 2, we will accurately discover which attributes this additional $.72 of passive return is due to. We add the appropriate portions of this $.72 to the period 2 attributes and we can then accurately discover the total dollar difference due to each attribute. This simple line of reasoning is the basis behind the Frongello method.

Note: This sequential linking style also assures the appropriate treatment of order dependence.