The Frongello Linking Algorithm

$$F_{tb} = G_{tb}.5[\prod_{j=1}^{t-1}(1+R_j)+\prod_{j=1}^{t-1}(1+\overline{R}_j)]+.5(R_t+\overline{R}_t)\sum_{j=1}^{t-1}F_{jb}$$

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The Linking Problem

Question 1 (Compounding total return):
If a portfolio earns 10% in period 1 and 12% in period 2, what is the return over the two periods?

Answer: 1.1*1.12-1= **23.2%**

This is called *compounding* and it is a widely accepted axiom of finance.

Question 2 (Linking absolute attribution):
Supposing further analysis of this portfolio revealed that the performance was attributed in the following manner (see below), what would be the contribution of bonds (stocks) over the two periods?

	Performance		
	Total	Due to Bonds	Due to Stocks
Period 1	10.0%	4.0%	6.0%
Period 2	12.0%	3.0%	9.0%
Total	23.2%	?	?

Answer (Solution in appendix):
Cumulative performance due to bonds = **7.39%**
Cumulative performance due to stocks = **15.81%**

Question 3 (Linking relative attribution):
To complicate the problem, what if the performance and attribution from problem 2 represented performance relative to some benchmark rather than the performance of the portfolio?

	Performance		Relative Performance		
	Portfolio	Benchmark	Difference	Due to Bonds	Due to Stocks
Period 1	15.0%	5.0%	10.0%	4.0%	6.0%
Period 2	22.0%	10.0%	12.0%	3.0%	9.0%
Total	40.3%	15.5%	24.8%	?	?

Notice that if the benchmark return were 0% in both periods, then problem 3 would be the same as problem 2. Including the benchmark return and doing a relative analysis adds further complexity to the problem. In this example we are now trying to explain 24.8% rather than the 23.2% in problem 2.

Answer (Solution in appendix):
Cumulative relative performance due to bonds = **7.94%**
Cumulative relative performance due to stocks = **16.86%**

Although numerous *approximations* to this solution were available in the mid 1990's, this exact solution, provided by the Frongello Linking Algorithm, was not available until the winter of 2002/2003. The rest of this paper will provide a brief chronology of events leading to the development of this algorithm. I will conclude with a discussion of the impact of this work.

Historical literature

The linking problem was a notorious challenge in finance. Over the years there have been many proposed solutions/approximations. While some have either inspired or laid the foundations for further study, others have provided little contribution to discovering a solution. In the interest of space I will only mention the works and people that have inspired my study of the subject.

Pre-1999
Linking is a prevalent problem in performance measurement groups across the globe. Most firms attempt to reengineer their single period formulas, some develop new long-term attribution methodologies, while others simply report the mysterious unexplained portion that arises during their multi-period analysis as an unexplained residual. My firm at this time was struggling with this problem also. I began to study the best available research.

Summer 1999
David Cariño proposes a very strong smoothing algorithm using logarithms. He proposes that; 1. Any strong algorithm should work with any single period attribution scheme, 2. A separate multi-period presentation format was unnecessary, and 3. Residuals were unacceptable. This was the best approximation to date.

Fall 2000
Jose Menchero develops a similar smoothing algorithm using Lagrange multipliers. He argues that the Cariño algorithm overweights periods of low returns. This is an interesting approximation and I curiously investigate his concerns.

Winter 2000/2001
Andre Mirabelli notes a critical flaw in the Cariño and Menchero approximations. He notes that one of the inputs of their smoothing scalars is the total performance of the cumulative period. This creates two key problems: 1. Future returns have to be known before a period can be scaled. 2. As additional return information becomes available, the smoothing scalars and corresponding performance history have to be restated every month.

Andre attempts to solve this problem by creating a new stepwise recursive method. He explains each month's contribution to relative performance in the month it occurs. Although Andre has great intentions and offers some very valuable insights, his results differ too much from the Cariño and Menchero approximations. I'm fascinated by his step-wise approach however and my study of the problem intensifies.

Fall 2001
Damien Laker argues that multiple-period attribution does not have an exact solution because multiple period attributes will not explain the total value added. I find his defense somewhat neglected and I'm fascinated that so many incredibly bright people can on the one hand be certain about single period results and on the other hand be convinced that multi-period certainty can not be achieved. I am determined to either find a solution or at least demonstrate why a solution is not possible.

Fall 2002
I'm at Taco Bell and although I should have been enjoying my burrito, I was feverishly scribbling on a napkin. After months of constant hypothesizing, I had what I thought was a working step-wise recursive solution to the problem. I rushed home to analyze the solution in front of my computer….a month later my first paper appeared in *the Journal of Performance Measurement*.

Impact

Shortly after the publication, I was asked by the Journal to defend some rather harsh criticism from Damien Laker. My second paper had a few more rigorous mathematical proofs and also some rather simple intuitive examples that were more popular with a wider audience. To my surprise a majority of the contributing authors to this issue submitted articles supporting my thesis. Furthermore, since my first publishing I have been cited 18 times in other published works and studied at numerous conferences.

My algorithm has been adopted by investment firms around the globe and I travel to speak at conferences around the country and abroad. I can trace the success of my method to the following factors

1. Although I am given credit for the discovery, my work was inspired and driven by the precedent setting philosophies of Cariño and Mirabelli. I tried to **preserve the strengths of these prior methods while addressing their weaknesses**.

2. I offered a solution with **intuitive appeal**. The simple algebraic reasoning behind the Frongello method is more palatable than the rather unintuitive formulas previously offered. Earlier I indirectly argued that the Menchero algorithm is a fair solution. But it is a little hard to swallow. Take a look.

$$(1/T)[(R-\overline{R})/((1+R)^{1/T}-(1+\overline{R})^{1/T})]+\frac{(R-\overline{R}-(1/T)[(R-\overline{R})/((1+R)^{1/T}-(1+\overline{R})^{1/T})]\sum_{j=1}^{T}(R_j-\overline{R}_j))(R_t-\overline{R}_t)}{\sum_{j=1}^{T}(R_j-\overline{R}_j)^2}$$

   Compare this to a simplified Frongello algorithm (only faint differences to my longer exact version seen on the title page, popularly used when the portfolio tracks the benchmark)

$$G_{tb}\prod_{j=1}^{t-1}(1+R_j)+\overline{R}_t\sum_{j=1}^{t-1}F_{jb}$$

3. Investment professionals do not like restating prior numbers. My recursive method **preserves historical results**.

4. **Daily attribution** has become the industry norm due to the correlation between measurement frequency and accuracy, the greater integration of risk analysis and performance measurement, and increased client/manager performance requests. These daily results can be easily linked with my algorithm.

I am excited my contributions have met such great acceptance. My work has contributed an enhancement to the investment management process. More transparent views into the performance of a portfolio help managers access and monitor risk and empower clients with an improved method for reviewing their fiduciaries.

I hope that the growing field of performance and attribution will someday make its way into the finance curriculum. Although there are numerous business schools that can teach an individual to manage a portfolio, without a solid understanding of performance and attribution these individuals and their constituents won't know if the investment manager is doing their job well.

Appendix

Frongello Linking Algorithm

$\sum_t \sum_b G_{tb} \neq R - \overline{R}$ Original attributes do not sum to cumulative value added

$\sum_t \sum_b F_{tb} = R - \overline{R}$ However, Frongello adjusted attributes do sum to cumulative value added

$$F_{tb} = G_{tb}.5[\prod_{j=1}^{t-1}(1+R_j) + \prod_{j=1}^{t-1}(1+\overline{R}_j)] + .5(R_t + \overline{R}_t)\sum_{j=1}^{t-1} F_{jb}$$

F_{tb} =Adjusted Attribute b in time period t
G_{tb} =Original Attribute b in time period t
R_t =Portfolio return in period t
$\overline{R}_t$ =Benchmark return in period t

Question 2 (Linking absolute attribution-solution):

Supposing further analysis of this portfolio revealed that the performance was attributed in the following manner, what would be the contribution of bonds (stocks) over the two periods?

	Performance		
	Total	Due to Bonds	Due to Stocks
Period 1	10.0%	4.0%	6.0%
Period 2	12.0%	3.0%	9.0%
Total	23.2%	?	?

Answer:
Cumulative performance due to bonds =
4%*.5*(1+0%+1+0%)+.5*(10%+0%)*0%
+ 3%*.5*(1+10%+1+0%)+.5*(12%+0%)*4% = **7.39%**

Cumulative performance due to stocks =
6%*.5*(1+0%+1+0%)+.5*(10%+0%)*0%
+ 9%*.5*(1+10%+1+0%)+.5*(12%+0%)*6% = **15.81%**

This result can be treated as a single period to link on a third period, etc.

Method Comparison:	Bonds	Stocks
Frongello Exact Solution	**7.390000%**	**15.810000%**
Cariño approximation (rounded)	**7.389730%**	**15.810270%**
Menchero approximation (rounded)	**7.386049%**	**15.813951%**

These differences compound geometrically with additional periods.

Question 3 (Linking relative attribution-solution):

To complicate the problem, what if the performance and attribution from problem 2 represented performance relative to some benchmark rather than the performance of the portfolio?

	Performance		Relative Performance		
	Portfolio	Benchmark	Difference	Due to Bonds	Due to Stocks
Period 1	15.0%	5.0%	10.0%	4.0%	6.0%
Period 2	22.0%	10.0%	12.0%	3.0%	9.0%
Total	40.3%	15.5%	24.8%	?	?

Notice that if the benchmark return were 0% in both periods, then problem 3 would be the same as problem 2. Including the benchmark return and doing a relative analysis adds further complexity to the problem. In this example we are now trying to explain 24.8% rather than the 23.2% in problem 2.

Answer:
Cumulative performance due to bonds =
4%*.5*(1+0%+1+0%)+.5*(15%+5%)*0%
+ 3%*.5*(1+15%+1+5%)+.5*(22%+10%)*4% = **7.94%**

Cumulative performance due to stocks =
6%*.5*(1+0%+1+0%)+.5*(15%+5%)*0%
+ 9%*.5*(1+15%+1+5%)+.5*(22%+10%)*6% = **16.86%**

This result can be treated as a single period to link on a third period, etc.

Method Comparison:	Bonds	Stocks
Frongello Exact Solution	**7.940000%**	**16.860000%**
Cariño approximation (rounded)	**7.939812%**	**16.860188%**
Menchero approximation (rounded)	**7.901670%**	**16.898330%**

These differences compound geometrically with additional periods.

Note (2006): I was not able to walk away with the Lemelson Prize. I believe the winner that year invented chocolate Legos. "Everyones favorite snack and toy!" How can anyone compete with that? Can't win 'em all.