

Linking Single Period Arithmetic Attribution Results

Arithmetic Attribution Definition

The difference between the portfolio and benchmark returns is explained by the sum of a set of attributes.

R = Portfolio Return

\bar{R} = Benchmark Return

$a + b + c + \dots + n$ = Attributes

$R - \bar{R} = a + b + c + \dots + n$

Portfolio return = 21%

Benchmark return = 11%

Allocation = 6%

Selection = 4%

21% - 11% = 6% + 4%

Challenge: Linking single period attribution results without an unexplained residual.

	Portfolio	Benchmark	Diff.	Allocation	Selection
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%
Total	37.94%	20.99%	=	16.95%	

Naïve Approaches:

Sum	Allocation	Selection	Total
	6.00%	4.00%	
	2.00%	3.00%	
	8.00%	7.00%	15.00%

15.00% ≠ 16.95%

Compound	Allocation	Selection	Total
	6.00%	4.00%	
	2.00%	3.00%	
	8.12%	7.12%	15.24%

15.24% ≠ 16.95%

Problem / Solution

Problem : Attributes can't be summed or compounded.

G_{tb} = Original attribute b in time t

$$\sum_t \sum_b G_{tb} \neq R - \bar{R}$$

$$\left[\prod_t \prod_b (1 + G_{tb}) \right] - 1 \neq R - \bar{R}$$

Solution : Adjust attributes so they can be summed.

F_{tb} = Adjusted attribute b in time t

$$\sum_t \sum_b F_{tb} = R - \bar{R}$$

Desirable Linking Algorithm Characteristics

(Cariño 1999)

Generality- The methodology should support any additive single period scheme.

Failing Example: Maribelli-Only links Brinson & Fachler attribution

Familiarity- The interpretation of the multi-period results should be the same as the single period results.

Failing Example: Laker-Cumulative results sacrifice sector level information

No Residuals/Distortions- The methodology should explain exactly the over/under performance without introducing unnecessary distortion.

Failing Examples: Kirievski-Residuals remain.

Campisi-Sign switching.

Algorithms that pass these criteria include:

Frongello, Modified Frongello, Cariño, and Menchero.

Coefficient Methods

$$F_{tb} = G_{tb} \text{ (Scaling Coefficient)}$$

$$\sum_t \sum_b F_{tb} = R - \bar{R}$$

Scaling Coefficient Calculations

Cariño(1999)

$$\frac{[\ln(1+R_t) - \ln(1+\bar{R}_t)] / (R_t - \bar{R}_t)}{[\ln(1+R) - \ln(1+\bar{R})] / (R - \bar{R})}$$

Menchero(2001)

$$(1/T)[(R - \bar{R}) / ((1+R)^{1/T} - (1+\bar{R})^{1/T})] + \frac{(R - \bar{R} - (1/T)[(R - \bar{R}) / ((1+R)^{1/T} - (1+\bar{R})^{1/T})]) \sum_{j=1}^T (R_j - \bar{R}_j)(R_t - \bar{R}_t)}{\sum_{j=1}^T (R_j - \bar{R}_j)^2}$$

Coefficient Solutions

	Portfolio	Benchmark	Diff.	Allocation	Selection
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%
Total	37.94%	20.99%	16.95%		

Menchero	Allocation	Selection	Coefficient	Adj. Alloc.	Adj. Selec.
Period 1	6.00%	4.00%	1.1286	6.77%	4.51%
Period 2	2.00%	3.00%	1.1329	2.27%	3.40%
Total				9.04%	7.91%

Cariño	Allocation	Selection	Coefficient	Adj. Alloc.	Adj. Selec.
Period 1	6.00%	4.00%	1.1152	6.69%	4.46%
Period 2	2.00%	3.00%	1.1597	2.32%	3.48%
Total				9.01%	7.94%

“Andrew, why do we need another algorithm?”

	Math Used	Approach
Menchero & Cariño	Lagrange Calculus & Natural Logarithms Graduate Level	Mathematics used to stretch known attributes until the unexplained voids are filled.
Andrew's Complaint	<ol style="list-style-type: none"> 1. Unintuitive 2. Confusing 3. No Critical Review 4. Unnecessary 	<ol style="list-style-type: none"> 1. Arbitrary 2. Fails to answer the question, “Where do the voids (residuals) come from?”
Andrew's Solution	Algebra High School Level	Identify the <u>causes</u> of these voids and attribute the voids to those causes.

Frongello Method is Based on Sound Assumptions

Adding \bar{R} to both sides of our attribution definition,

$$R - \bar{R} = a + b + c + \dots + n$$

we arrive at a definition of portfolio return.

$$R = \bar{R} + a + b + c + \dots + n$$

Portfolio return = 21%

Benchmark return = 11%

Allocation = 6%

Selection = 4%

$$21\% = 11\% + 6\% + 4\%$$

Frongello Dollar Example

	Port.	Bench.	Diff.	Alloc.	Select.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%
Total	37.94%	20.99%	16.95%		

Start Value for Portfolio and Benchmark=\$100

Expectations:

Portfolio Dollar return = $\$100 * 37.94\% = \37.94

Benchmark Dollar return = $\$100 * 20.99\% = \20.99

Difference = $\$37.94 - \$20.99 = \$16.95$

The \$16.95 difference comes from allocation and selection.

Frongello Example – First Adjustment

	Portfolio Return			Benchmark Return
	Bench.	Alloc.	Select.	Bench.
Period 1	11.00%	6.00%	4.00%	11.00%
Period 2	9.00%	2.00%	3.00%	9.00%

	Portfolio					Benchmark		
	Beg MV	Bench	Alloc.	Select.	End MV	Beg MV	Bench	End MV
Period 1	\$100.00	\$11.00	\$6.00	\$4.00	\$121.00	\$100.00	\$11.00	\$111.00
Period 2	\$121.00	\$10.89	\$2.42	\$3.63	\$137.94	\$111.00	\$9.99	\$120.99

Portfolio dollar return = $\$137.94 - \$100.00 = \$37.94$

Benchmark dollar return = $\$120.99 - \$100.00 = \$20.99$

We are expecting \$37.94 and \$20.99.

How much of the \$16.95 difference comes from Allocation and Selection?

Frongello Example – Second Adjustment

	Portfolio Return			Benchmark Return
	Bench.	Alloc.	Select.	Bench.
Period 1	11.00%	6.00%	4.00%	11.00%
Period 2	9.00%	2.00%	3.00%	9.00%

	Portfolio					Benchmark		
	Beg MV	Bench	Alloc.	Select.	End MV	Beg MV	Bench	End MV
Period 1	\$100.00	\$11.00	\$6.00	\$4.00	\$121.00	\$100.00	\$11.00	\$111.00
Period 2	\$121.00	\$10.89	\$2.42	\$3.63	\$137.94	\$111.00	\$9.99	\$120.99

Looking for \$16.95 of outperformance

Total Allocation = \$6.00 + \$2.42 = \$8.42

Total Selection = \$4.00 + \$3.63 = \$7.63

\$8.42 + \$7.63 = \$16.05

Allocation & Selection explain only \$16.05 out of \$16.95

Where is this additional \$.90 coming from?

Frongello Example – Second Adjustment

	Portfolio Return			Benchmark Return
	Bench.	Alloc.	Select.	Bench.
Period 1	11.00%	6.00%	4.00%	11.00%
Period 2	9.00%	2.00%	3.00%	9.00%

	Portfolio					Benchmark		
	Beg MV	Bench	Alloc.	Select.	End MV	Beg MV	Bench	End MV
Period 1	\$100.00	\$11.00	\$6.00	\$4.00	\$121.00	\$100.00	\$11.00	\$111.00
Period 2	\$121.00	\$10.89	\$2.42	\$3.63	\$137.94	\$111.00	\$9.99	\$120.99

In period 2, the portfolio earned \$.90 more at the benchmark rate of return.

How?

Because the portfolio base is \$10 larger than the benchmark base.

$$\$10 * 9\% = \$.90$$

Frongello Example – Second Adjustment

	Portfolio Return			Benchmark Return
	Bench.	Alloc.	Select.	Bench.
Period 1	11.00%	6.00%	4.00%	11.00%
Period 2	9.00%	2.00%	3.00%	9.00%

	Portfolio					Benchmark		
	Beg MV	Bench	Alloc.	Select.	End MV	Beg MV	Bench	End MV
Period 1	\$100.00	\$11.00	\$6.00	\$4.00	\$121.00	\$100.00	\$11.00	\$111.00
Period 2	\$121.00	\$10.89	\$2.42	\$3.63	\$137.94	\$111.00	\$9.99	\$120.99

The extra \$10 comes from allocation (\$6) and selection (\$4) in period 1.

The additional \$.90 comes from these attributes earning the benchmark rate.

$$(\$6 + \$4) * 9\% = \$.54 + \$.36$$

Frongello Example – Both Adjustments

	Portfolio Return			Benchmark Return
	Bench.	Alloc.	Select.	Bench.
Period 1	11.00%	6.00%	4.00%	11.00%
Period 2	9.00%	2.00%	3.00%	9.00%

	Portfolio					Benchmark		
	Beg MV	Bench	Alloc.	Select.	End MV	Beg MV	Bench	End MV
Period 1	\$100.00	\$11.00	\$6.00	\$4.00	\$121.00	\$100.00	\$11.00	\$111.00
Period 2	\$121.00	\$9.99	\$2.96	\$3.99	\$137.94	\$111.00	\$9.99	\$120.99

Reallocate the \$.90 from the benchmark return to the attributes.

$$\text{Allocation} = \$6 + \$2.42 + \$.54 = \$8.96$$

$$\text{Selection} = \$4 + \$3.63 + \$.36 = \$7.99$$

$$\$8.96 + \$7.99 = \$16.95$$

Finally, we have attributed the exact amount we are trying to explain!

Frongello Adjustments Recap

Before we added attributes, we made two adjustments.

Adjustment 1:

Scale the current attribute by the total portfolio return through the prior period.

Adjustment 2:

Multiply the prior attributes by the current benchmark return.

Percent Example

	Portfolio	Benchmark	Diff.	Allocation	Selection
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%
Total	37.94%	20.99%	16.95%		

Frongello Solution:

	Allocation	Selection
Period 1	6.00%	4.00%
Period 2 (Adj. 1)	$2\% \times 1.21 = 2.42\%$	$3\% \times 1.21 = 3.63\%$
Period 2 (Adj. 2)	$6\% \times 9\% = .54\%$	$4\% \times 9\% = .36\%$
Total	8.96%	7.99%

Furthermore:

- Treat the 2 period result as a single period, and link on a third, etc.

Frongello - Multiple Period Example

	Port.	Bench.	Diff.	Alloc.	Select.	Adj. Alloc.	Adj. Selec.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	2.96%	3.99%
Period 3	20.00%	12.00%	8.00%	1.00%	7.00%	2.45%	10.61%
Period 4	17.00%	10.00%	7.00%	5.00%	2.00%	9.42%	5.17%
Total	93.67%	49.06%	44.61%			20.83%	23.78%

Selection Illustrated					
	Current Attribute	Port Ret Thru n-1	Current Bench Ret	Sum Prior Adj Attributes	Frongello Adjusted Attribute*
Period 1	4.00%	0.00%	11.00%	0.00%	4.00%
Period 2	3.00%	21.00%	9.00%	4.00%	3.99%
Period 3	7.00%	37.94%	12.00%	7.99%	10.61%
Period 4	2.00%	65.53%	10.00%	18.60%	5.17%

* = Curr Attribute x (1+Port Ret Thru n-1) + Curr Bench Ret x Sum Prior Adj Attributes

- The formula reduces to the Frongello algorithm

Frongello Adjusted Attributes

$$F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \bar{R}_t \sum_{j=1}^{t-1} F_{jb}$$

Intuitive interpretation:

Each **original attribute** is scaled by the **portfolio total return through the prior period** and the **current period return of the benchmark** compounds with the **total return due to that attribute through the prior period**.

Differentiating Characteristics

(Mirabelli 2000)

Non A-Causal - The linking methodology should not be dependent on future events when scaling single period results.

(Frongello 2002)

Sincerity - The method should reflect the reality of fundamental financial principles. Beware of mathematical rhetoric.

Intuitive - The method should preferably use mathematics friendly to a wide audience.

Order Dependence - The ordering of periods will affect cumulative attribution results when defining the portfolio investment base by total return. By definition accurate, despite some protests.

Return Sensitive – Periods of low returns will require higher scaling than periods of high returns, and vice versa.

A-Causality - Frongello vs. Menchero

The Frongello method is not dependent on future returns.
 Prior period scaling does not change when adding periods.

<u>Frongello</u>	Port.	Bench.	Diff.	Alloc.	Select.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.96%	3.99%
Total	37.94%	20.99%	16.95%	8.96%	7.99%

<u>Frongello</u>	Port.	Bench.	Diff.	Alloc.	Select.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.96%	3.99%
Period 3	20.00%	12.00%	8.00%	2.45%	10.61%
Total	65.53%	35.51%	30.02%	11.41%	18.60%

The Menchero scaling coefficient is dependent on future returns.
 Prior period scaling does change when adding periods.

<u>Menchero</u>	Port.	Bench.	Diff.	Alloc.	Select.
Period 1	21.00%	11.00%	10.00%	6.77%	4.51%
Period 2	14.00%	9.00%	5.00%	2.27%	3.40%
Total	37.94%	20.99%	16.95%	9.04%	7.91%

<u>Menchero</u>	Port.	Bench.	Diff.	Alloc.	Select.
Period 1	21.00%	11.00%	10.00%	7.82%	5.22%
Period 2	14.00%	9.00%	5.00%	2.61%	3.92%
Period 3	20.00%	12.00%	8.00%	1.31%	9.14%
Total	65.53%	35.51%	30.02%	11.74%	18.28%

Sincerity - Frongello vs. Cariño

A model should attribute the contribution to excess return in the period in which it occurs.

	Port.	Bench.	Cum. Port	Cum. Bench.	Cum. Diff.	Cont. Cum. Diff.
Period 1	21.00%	11.00%	21.00%	11.00%	10.00%	10.00%
Period 2	14.00%	9.00%	37.94%	20.99%	16.95%	6.95%
Period 3	20.00%	12.00%	65.53%	35.51%	30.02%	13.07%

Frongello Method:

Contribution to excess return is attributed to the period in which it occurs.

Frongello	Port.	Bench.	Diff.	Adj. Alloc.	Adj. Select.	Cont. Cum. Attr.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	10.00%
Period 2	14.00%	9.00%	5.00%	2.96%	3.99%	6.95%
Period 3	20.00%	12.00%	8.00%	2.45%	10.61%	13.07%
Total	65.53%	35.51%	30.02%	11.41%	18.60%	

Cariño Method:

Contribution to excess return is **not** attributed to the period in which it occurs.

Cariño	Port.	Bench.	Diff.	Adj. Alloc.	Adj. Select.	Cont. Cum. Attr.
Period 1	21.00%	11.00%	10.00%	7.76%	5.18%	12.94%
Period 2	14.00%	9.00%	5.00%	2.69%	4.04%	6.73%
Period 3	20.00%	12.00%	8.00%	1.29%	9.06%	10.35%
Total	65.53%	35.51%	30.02%	11.75%	18.27%	

Return Sensitivity – Cariño vs. Menchero

Cariño	Port	Bench	Diff.	Alloc	Selec	Coef	Adj. Alloc	Adj. Selec
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	1.17	6.99%	4.66%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	1.21	2.42%	3.64%
Period 3	8.00%	1.00%	7.00%	1.00%	6.00%	1.29	1.29%	7.76%
Total	48.98%	22.20%	26.78%				10.71%	16.06%

Menchero	Port	Bench	Diff.	Alloc	Selec	Coef	Adj. Alloc	Adj. Selec
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	1.22	7.29%	4.86%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	1.22	2.44%	3.66%
Period 3	8.00%	1.00%	7.00%	1.00%	6.00%	1.22	1.22%	7.31%
Total	48.98%	22.20%	26.78%				10.95%	15.83%

Notice that the Cariño scaling coefficients vary with the level of return while the Menchero coefficients do not.

Attributes are a component of total return and they compound with the growth occurring in other periods.

Therefore, an attribute in a lower return period should be scaled more than a comparable attribute in a higher return period.

Frongello methods agree with Cariño method on this issue.

Order Dependence - Frongello

BEFORE	Port.	Bench.	Diff.	Alloc.	Select.	Adj. Alloc.	Adj. Selec.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	2.96%	3.99%
Period 3	8.00%	1.00%	7.00%	1.00%	6.00%	1.47%	8.36%
Total	48.98%	22.20%	26.78%			10.43%	16.35%

AFTER	Port.	Bench.	Diff.	Alloc.	Select.	Adj. Alloc.	Adj. Selec.
Period 1	8.00%	1.00%	7.00%	1.00%	6.00%	1.00%	6.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	2.25%	3.78%
Period 3	21.00%	11.00%	10.00%	6.00%	4.00%	7.74%	6.00%
Total	48.98%	22.20%	26.78%			10.99%	15.78%

Reversing the periods produces a different result!

Don't read this page unless you are a fanatic!!

If you define investment base by	and compound past attributes by	the cross product of Selection earned over Allocation will be assigned to	Use the following formula:
Portfolio Total Return	Index Total Return	Selection	<p>Frongello Linking Algorithm</p> $F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \bar{R}_t \sum_{j=1}^{t-1} F_{jb}$
Index Total Return	Portfolio Total Return	Allocation	<p>Reversed Frongello Linking Algorithm</p> $F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + \bar{R}_j) + R_t \sum_{j=1}^{t-1} F_{jb}$
Average Total Return	Average Total Return	Half Selection & Half Allocation	<p>Modified Frongello Linking Algorithm</p> $F_{tb} = G_{tb} \cdot 5 \left[\prod_{j=1}^{t-1} (1 + R_j) + \prod_{j=1}^{t-1} (1 + \bar{R}_j) \right] + .5(R_t + \bar{R}_t) \sum_{j=1}^{t-1} F_{jb}$

Frongello

Frongello	Port	Bench	Diff.	Alloc	Selec	Adj. Alloc	Adj. Selec
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	2.96%	3.99%
Total	48.98%	22.20%	26.78%			8.96%	7.99%

$$F_{tb} = G_{tb} \prod_{j=1}^{t-1} (1 + R_j) + \bar{R}_t \sum_{j=1}^{t-1} F_{jb}$$

Modified Frongello

M. Frongello	Port	Bench	Diff.	Alloc	Selec	Adj. Alloc	Adj. Selec
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	3.01%	3.94%
Total	48.98%	22.20%	26.78%			9.01%	7.94%

$$F_{tb} = G_{tb} .5 \left[\prod_{j=1}^{t-1} (1 + R_j) + \prod_{j=1}^{t-1} (1 + \bar{R}_j) \right] + .5 (R_t + \bar{R}_t) \sum_{j=1}^{t-1} F_{jb}$$

Modified Frongello - Multiple Period Example

	Port.	Bench.	Diff.	Alloc.	Select.	Adj. Alloc.	Adj. Selec.
Period 1	21.00%	11.00%	10.00%	6.00%	4.00%	6.00%	4.00%
Period 2	14.00%	9.00%	5.00%	2.00%	3.00%	3.01%	3.94%
Period 3	20.00%	12.00%	8.00%	1.00%	7.00%	2.74%	10.33%
Period 4	17.00%	10.00%	7.00%	5.00%	2.00%	9.11%	5.48%
Total	93.67%	49.06%	44.61%			20.86%	23.75%

Selection Illustrated					
	Current Attribute	Avg. Cum. Ret Thru n-1	Current Avg. Ret	Sum Prior Adj Attributes	Mod - Frongello Adjusted Attribute*
Period 1	4.00%	0.00%	16.00%	0.00%	4.00%
Period 2	3.00%	16.00%	11.50%	4.00%	3.94%
Period 3	7.00%	29.47%	16.00%	7.94%	10.33%
Period 4	2.00%	50.52%	13.50%	18.27%	5.48%

$$= \text{Curr Attribute} \times \left[\frac{(1 + \text{Port Ret Thru } n-1) + (1 + \text{Bench Ret Thru } n-1)}{2} + \frac{[\text{Curr Port Ret} + \text{Curr Bench Ret}]}{2} \times \text{Sum Prior Adj Attributes} \right]$$

- The formula reduces to the Modified Frongello algorithm

Do these methods provide different answers?

Recent Frongello Study

Periods=120 (Monthly, Sep 93 – Aug 03)

Trials = 10,000

Index= Lehman Brothers Aggregate

Portfolio = Index carve outs (100 randomly selected issues)

Scheme = Duration, Allocation, Selection by sector buckets

Results

Cariño and Modified Frongello linked results are almost identical.

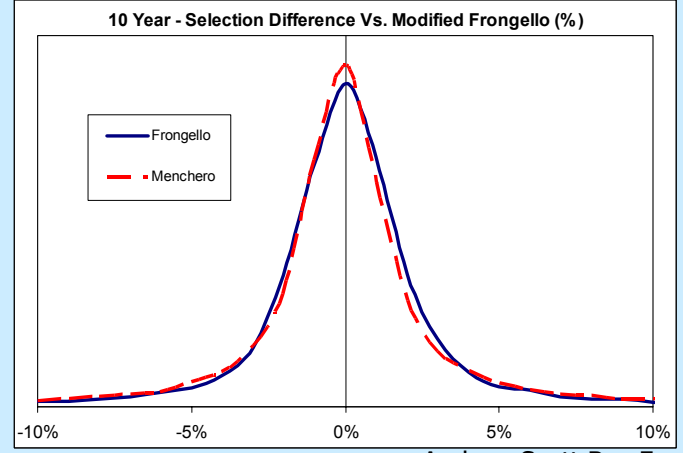
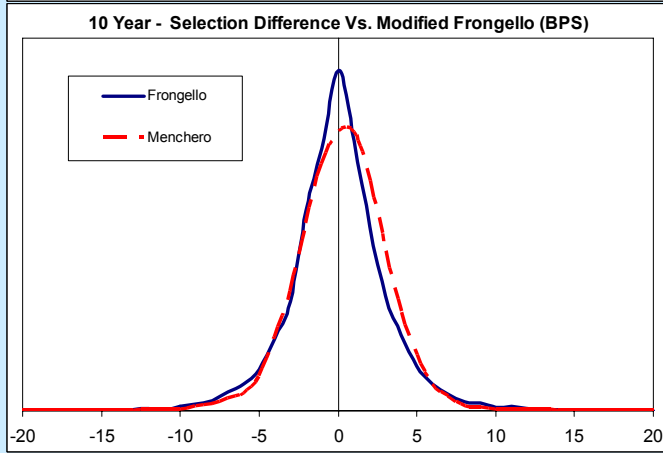
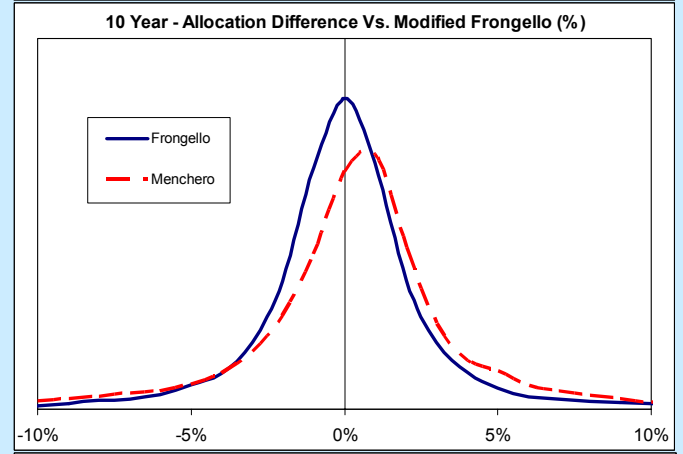
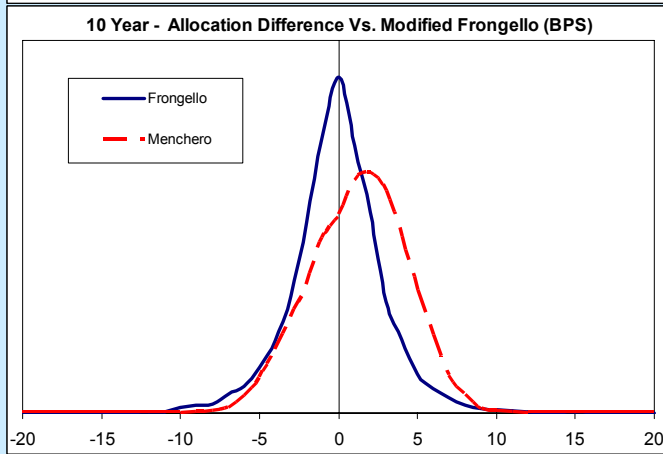
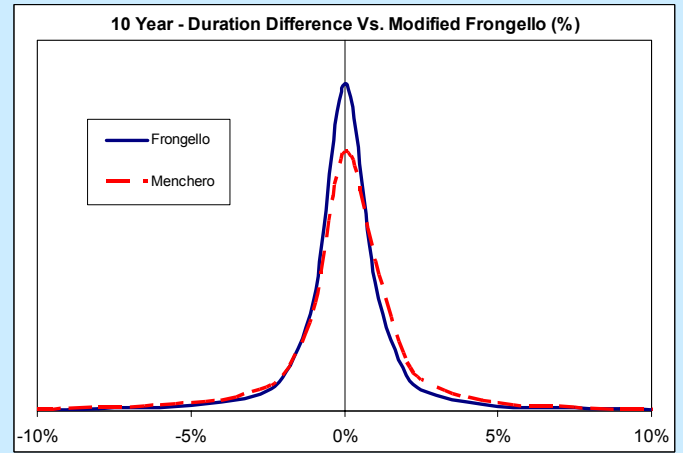
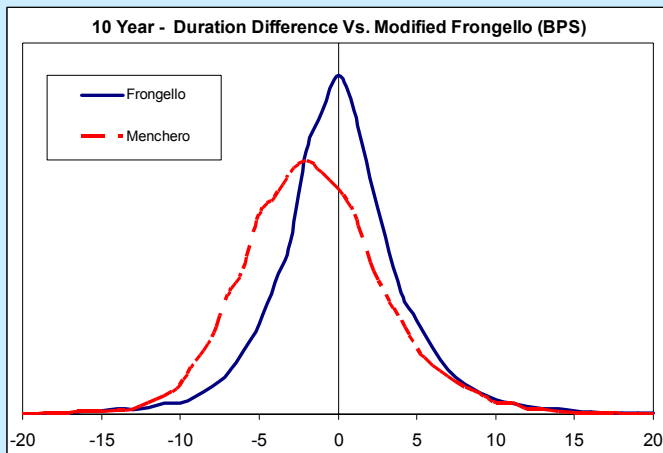
Maximum BPS difference in linked results never more than 1/3 of a BP for any attribute.

Roughly 97% of the time the linked attributes differ by less than .1%.





























Frongello and Menchero provide very close approximations to Modified Frongello & Cariño.

Conclusion

The linked results are *mathematically* different between the methods, but these differences are not *materially* different.



Linking method comparison

	Non A-Causal	Intuitive	Non Order Dependent	Sincerity	Return Sensitive	Suitable for Absolute Attribution	Agreement Among Methods
Modified Frongello							
Frongello							
Cariño							
Menchero							

Questions?